Uniform analyticity and exponential decay of the semigroup associated with a thermoelastic plate equation with perturbed boundary conditions

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Overview

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- Problem formulation.

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- Some literature.

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- Proof of Theorem.

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Let Ω be a bounded open subset of $\mathbb{R}^N$, with smooth enough boundary Γ. Let ν denote the unit outward normal to Γ. Let $\gamma \in (0, \infty)$ be a parameter. Consider the perturbed thermoelastic plate equations:

$$\begin{cases} y_{\gamma,tt} + \Delta^2 y_\gamma + \alpha \Delta \theta_\gamma = 0 \text{ in } \Omega \times (0, \infty) \\ \theta_{\gamma,t} - \kappa \Delta \theta_\gamma - \beta \Delta y_{\gamma,t} = 0 \text{ in } \Omega \times (0, \infty) \\ y_\gamma = 0, \quad \gamma \Delta y_\gamma + \partial_\nu y_\gamma = 0, \quad \theta_\gamma = 0 \text{ on } \Sigma = \Gamma \times (0, \infty) \\ y_\gamma(0) = y_\gamma^0 \in V, \quad y_t(0) = y^1 \in H, \quad \theta_\gamma(0) = \theta^0 \in H, \end{cases}$$

where $V = H^2(\Omega) \cap H_0^1(\Omega), \quad H = L^2(\Omega)$. Set $W = H_0^1(\Omega)$.
Introduce the Hilbert space over the field of complex numbers $\mathcal{H}_\gamma = V \times H \times H$ equipped with the norm:

$$||(u, v, w)||_\gamma^2 = \int_\Omega \left\{ |\Delta u|^2 + |v|^2 + \frac{\alpha}{\beta} |w|^2 \right\} dx + \frac{1}{\gamma} \int_\Gamma |\partial_\nu u|^2 \, d\Gamma.$$

Setting $Z_\gamma = \begin{pmatrix} y_\gamma \\ y_{\gamma,t} \\ \theta \end{pmatrix}$, our system may be recast as:

$$\dot{Z}_\gamma - \mathcal{A}_\gamma Z_\gamma = 0 \text{ in } (0,\infty),$$
$$Z_\gamma(0) = \begin{pmatrix} y_\gamma^0 \\ y^1 \\ \theta^0 \end{pmatrix},$$

where the unbounded operator $\mathcal{A}_\gamma$ is given by

$$\mathcal{A}_\gamma = \begin{pmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & -\alpha\Delta \\ 0 & \beta\Delta & \kappa\Delta \end{pmatrix}$$

$$D(\mathcal{A}_\gamma) = \Big\{(u, v, w) \in V \times V \times W;\ \Delta^2 u + \alpha \Delta w \in L^2(\Omega),$$
$$\beta \Delta v + \kappa \Delta w \in L^2(\Omega);\ \gamma \Delta u + \partial_\nu u = 0 \text{ on } \Gamma\Big\}$$
$$= \Big\{(u, v, w) \in (H^4(\Omega) \cap V) \times V \times (H^2(\Omega) \cap W);$$
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The questions to be addressed are the following:

- Does the operator $\mathcal{A}_\gamma$ generate a $\mathcal{C}_0$ semigroup?

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- Is that semigroup uniformly exponentially stable with respect to γ?

Main Theorem

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For every $\gamma > 0$, the linear operator $\mathcal{A}_\gamma$ generates a $\mathcal{C}_0$-semigroup of contractions $(S_\gamma(t))_{t\geq 0}$ that is uniformly, with respect to γ, analytic: there exists a positive constant K independent of γ such that for every $t > 0$, one has:

$$||\mathcal{A}_\gamma S_\gamma(t)Z^0||_\gamma \leq \frac{K||Z^0||_\gamma}{t}, \quad \forall Z^0 \in \mathcal{H}_\gamma, \quad \forall \gamma > 0.$$

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Furthermore, for each $\gamma > 0$, the semigroup $(S_\gamma(t))_{t\geq 0}$ is uniformly exponentially stable; more precisely there exist positive constants M and λ that are independent of γ such that for every $t \geq 0$, one has:

$$||S_\gamma(t)Z^0||_\gamma \leq M\exp(-\lambda t)||Z^0||_\gamma, \quad \forall Z^0 \in \mathcal{H}_\gamma, \quad \forall \gamma > 0.$$

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All those works are closely connected to Lagnese book on the stabilization of thin plates [SIAM, 1989].

One can show that as $\gamma \to \infty$, the unique weak solution of the perturbed system converges in a suitable sense to the unique weak solution of the system

$$\begin{cases} y_{tt} + \Delta^2 y + \alpha\Delta\theta = 0 \text{ in } \Omega \times (0, \infty) \\ \theta_t - \kappa\Delta\theta - \beta\Delta y_t = 0 \text{ in } \Omega \times (0, \infty) \\ y = 0, \quad \Delta y = 0, \quad \theta = 0 \text{ on } \Sigma = \Gamma \times (0, \infty) \\ y(0) = y^0 \in V, \quad y_t(0) = y^1 \in H, \quad \theta(0) = \theta^0 \in H. \end{cases}$$

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- Earlier results about the exponential decay and the analyticity of the semigroup in the case of hinged or clamped boundary conditions [Kim, SIMA 1992, Liu-Renardy, AML 1995] may be recovered by either letting γ go to infinity or zero; this is feasible because our proof below clearly shows that the constants in our estimates are independent of γ.
- We improve on the result in Liu-Renardy in the case of clamped boundary conditions, as we work in the usual functional space $H_0^2(\Omega) \times [L^2(\Omega)]^2$ while those authors worked in an *ad hoc* functional space $X = \{u \in L^2(\Omega); \Delta u = 0\}^{\perp} \times [L^2(\Omega)]^2$.

- We also note that the system considered in Lasiecka-Triggiani [Abst.App.An., 1998] is very close to the system being discussed; however their approach for proving the analyticity is different from ours; in particular given that they use the usual norm on $H^2(\Omega)$, their constants would explode as γ goes to zero. Besides, as we shall see below, the desire to establish estimates that are uniform in γ involves more technicalities than otherwise.

First, we shall prove that for each $\gamma > 0$, the unbounded operator $\mathcal{A}_\gamma$ generates a $\mathcal{C}_0$ semigroup of contractions $(S_\gamma(t))_{t\geq 0}$.
We have:

- the operator $\mathcal{A}_\gamma$ is dissipative as:

$$\Re\left(\mathcal{A}_\gamma Z, Z\right) = -\frac{\alpha\kappa}{\beta}\int_\Omega |\nabla w|^2\,dx \leq 0, \quad \forall Z = (u, v, w) \in \mathcal{D}(\mathcal{A}_\gamma).$$

- $\mathcal{I} - \mathcal{A}_\gamma$ is onto, by Lax-Milgram Lemma, ($\mathcal{I}$ denotes the identity operator).

Consequently, the operator $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$ by Lumer-Phillips Theorem [Pazy, 1983]. Since $D(\mathcal{A}_\gamma)$ is dense in $\mathcal{H}_\gamma$, one checks that $\mathcal{A}_\gamma$ has a compact resolvent; therefore its spectrum is discrete.
Further $\sigma(\mathcal{A}_\gamma) \cap i\mathbb{R} = \emptyset$.

Now, according to Theorem 5.2 in Chap. 2 of Pazy book, for analyticity, and Theorem 3 in [Huang, 1985] or Corollary 4 in [Prüss, 1984], for exponential stability, it remains to show that there exists a positive constant C_0 independent of γ such that

$$\sup\{||b(ib-\mathcal{A}_\gamma)^{-1}||_{\mathcal{L}(\mathcal{H}_\gamma)};\ b\in\mathbb{R}\}\leq C_0,$$

and

$$\sup\{||(ib-\mathcal{A}_\gamma)^{-1}||_{\mathcal{L}(\mathcal{H}_\gamma)};\ b\in\mathbb{R}\}\leq C_0.$$

To prove those estimates, it's enough to show that there exists $C_0 > 0$ such that for every $U \in \mathcal{H}_\gamma$, one has:

$$||b(ib-\mathcal{A}_\gamma)^{-1}U||_\gamma + ||(ib-\mathcal{A}_\gamma)^{-1}U||_\gamma \le C_0||U||_\gamma, \quad \forall b \in \mathbb{R}, \quad \forall \gamma > 0.$$

Let $b \in \mathbb{R}$, $U = (f, g, h) \in \mathcal{H}_\gamma$, and let $Z = (u, v, w) \in D(\mathcal{A}_\gamma)$ such that

$$(ib-\mathcal{A}_\gamma)Z = U. \tag{1}$$

Multiply both sides of that equation by Z, then take the real part of the inner product in $\mathcal{H}_\gamma$ to derive:

$$\frac{\alpha\kappa}{\beta}\int_\Omega |\nabla w|^2\,dx = \Re(U, Z) \le ||U||_\gamma ||Z||_\gamma.$$

Equation (**??**) may be rewritten:

$$\begin{aligned}
&ibu - v = f\\
&ibv + \Delta^2 u + \alpha\Delta w = g\\
&ibw - \beta\Delta v - \kappa\Delta w = h\\
&u = 0, \quad \gamma\Delta u + \partial_\nu u = 0, \quad v = 0, \quad w = 0 \text{ on } \Gamma.
\end{aligned}$$

(2)

The desired estimate will be established if we show the following estimate:

$$(|b|+1)||Z||_\gamma \le C_0||U||_\gamma, \quad \forall\gamma>0, \quad \forall b\in\mathbb{R}.$$

Step 1. In this step, we are going to show that for every $\varepsilon>0$, there exists a positive constant C_ε, independent of γ and b such that

$$||Z||_\gamma \le \varepsilon|b|||Z||_\gamma + C_\varepsilon||U||_\gamma.$$

Multiply the first equation in (**??**) by $\bar{u}$, apply Green's formula, take the real parts, then use Hölder inequality to derive

$$\begin{aligned}|\Delta u|_2^2+\frac{1}{\gamma}\int_\Gamma|\partial_\nu u|^2\,d\Gamma &= \Re\int_\Omega\{v(\bar{v}+\bar{f})-\alpha w\Delta\bar{u}+g\bar{u}\}\,dx\\ &\le |v|_2^2+C_0(||Z||_\gamma||U||_\gamma+||U||_\gamma^{\frac{1}{2}}||Z||_\gamma^{\frac{3}{2}}).\end{aligned}$$

Step 1 Cont'ed

If G denotes the inverse of the operator $-\Delta$ with Dirichlet boundary conditions, multiply the third equation in (**??**) by $G\bar{v}$, apply Green's formula, take the real parts, then use Hölder inequality, Young inequality to obtain

$$2|v|_2^2 = \Re \int_\Omega \{-ib\bar{v}Gw - \kappa w\bar{v} + \bar{v}Gh\}\, dx$$
$$\leq \varepsilon^2|b|^2|v|_2^2 + C_\varepsilon(||Z||_\gamma||U||_\gamma + ||U||_\gamma^{\frac{1}{2}}||Z||_\gamma^{\frac{3}{2}}), \quad \forall\varepsilon > 0,$$

where, here and in the sequel, C_ε is a generic positive constant independent of γ and b.
Hence

$$||Z||_\gamma \leq \varepsilon|b|||Z||_\gamma|| + C_\varepsilon(||Z||_\gamma||U||_\gamma + ||U||_\gamma^{\frac{1}{2}}||Z||_\gamma^{\frac{3}{2}}), \quad \forall\varepsilon > 0.$$

Step 2

Here, we will show that the following estimate holds

$$|b||w|_2 \leq \varepsilon |b|||Z||_\gamma + C_\varepsilon ||U||_\gamma, \quad \forall b \neq 0. \tag{3}$$

Set $w = w_1 + w_2$, where $w_1 \in W$ and $w_2 \in H$, with

$$ibw_1 - \Delta w_1 = h, \quad ibw_2 = \kappa \Delta w + \beta \Delta v - \Delta w_1.$$

Proceeding as above, one easily derives from the left equation

$$|b||w_1|_2 + |b|^{\frac{1}{2}}||w_1||_W + |\Delta w_1|_2 \leq C_0 ||U||_\gamma.$$

On the other hand, it follows from the right equation

$$|b|||w_2||_{H^{-2}(\Omega)} \leq C_0(|w|_2 + |v|_2 + |w_1|_2) \leq C_0(||Z||_\gamma + |b|^{-1}||U||_\gamma).$$

Step 2 Cont'ed

Now, by Lions' interpolation inequality as well as the last two estimates, and the fact that $||w_2||_W \leq ||w||_W + ||w_1||_W$, we derive

$$
\begin{aligned}
|b||w_2|_2 &\leq C_0|b|||w_2||^{\frac{1}{3}}_{H^{-2}(\Omega)}||w_2||^{\frac{2}{3}}_{H^1(\Omega)} \\
&\leq C_0 b^{\frac{2}{3}}(||Z||_\gamma + |b|^{-1}||U||_\gamma)^{\frac{1}{3}}(||U||^{\frac{1}{2}}_\gamma ||Z||^{\frac{1}{2}}_\gamma + |b|^{\frac{-1}{2}}||U||_\gamma)^{\frac{2}{3}}.
\end{aligned}
$$

Step 3.

Here, we shall prove:

$$|b||v|_2 \leq \varepsilon|b|||Z||_\gamma + C_\varepsilon||U||_\gamma, \quad \forall b \neq 0. \tag{4}$$

For the sequel, we also need to estimate $|b|||w_2||_{H^{-1}(\Omega)}$. Applying Lions' interpolation inequality once more and proceeding as above, one gets

$$\begin{aligned} |b|||w_2||_{H^{-1}(\Omega)} &\leq C_0|b|||w_2||^{\frac{2}{3}}_{H^{-2}(\Omega)}||w_2||^{\frac{1}{3}}_{H^1(\Omega)} \\ &\leq C_0|b|^{\frac{1}{3}}(||Z||_\gamma + |b|^{-1}||U||_\gamma)^{\frac{2}{3}}(||U||^{\frac{1}{2}}_\gamma||Z||^{\frac{1}{2}}_\gamma + |b|^{\frac{-1}{2}}||U||_\gamma)^{\frac{1}{3}}. \end{aligned}$$

Step 3 Cont'ed

Set $v = v_1 + v_2$, where $v_1 \in V$ and $v_2 \in H$, with

$$ibv_1 - \Delta v_1 = g, \quad ibv_2 = -\Delta^2 u - \alpha\Delta w - \Delta v_1.$$

One checks the following estimates:

$$|b||v_1|_2 + |b|^{\frac{1}{2}}||v_1||_W + |\Delta v_1|_2 \leq C_0||U||_\gamma,$$

and

$$|b|||v_2||_{H^{-2}()} \leq C_0(|\Delta u|_2 + |w|_2 + |v_1|_2) \leq C_0(||Z||_\gamma + |b|^{-1}||U||_\gamma).$$

Now, using the heat equation:

$$ibw - \kappa\Delta w - \beta\Delta v = h,$$

one can show:

$$||v||_W \leq C_0(||U||_\gamma^{\frac{1}{2}}||Z||_\gamma^{\frac{1}{2}} + ||U||_\gamma^{\frac{1}{2}}) + C_0|b|||w_2||_{H^{-1}(\Omega)}.$$

Step 3 Cont'ed

Applying Lions' interpolation inequality once more, and using the fact that $||v_2||_W \leq ||v_1||_W + ||v||_W$, we find:

$$|b||v_2|_2 \leq C_0|b|||v_2||^{\frac{1}{3}}_{H^{-2}(\Omega)}||v_2||^{\frac{2}{3}}_{H^1()}$$
$$\leq C_0 b^{\frac{2}{3}}(||Z||_\gamma + |b|^{-1}||U||_\gamma)^{\frac{1}{3}}(|b|^{\frac{-1}{2}}||U||_\gamma + ||v||_W)^{\frac{2}{3}}.$$

Combine this estimate with the estimate:

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Combine this estimate with the estimate:

$$||v||_W \leq C_0(||U||^{\frac{1}{2}}_\gamma||Z||^{\frac{1}{2}}_\gamma + ||U||^{\frac{1}{2}}_\gamma) + C_0|b|||w_2||_{H^{-1}(\Omega)}.$$

to derive:

$$|b||v_2|_2 \leq C_0(|b|^{\frac{1}{3}}||Z||^{\frac{1}{3}}_\gamma||U||^{\frac{2}{3}}_\gamma + |b|^{\frac{2}{3}}||Z||^{\frac{2}{3}}_\gamma||U||^{\frac{1}{3}}_\gamma + |b|^{\frac{8}{9}}||Z||^{\frac{8}{9}}_\gamma||U||^{\frac{1}{9}}_\gamma$$
$$+ |b|^{\frac{7}{9}}||Z||^{\frac{7}{9}}_\gamma||U||^{\frac{2}{9}}_\gamma + |b|^{\frac{5}{9}}||Z||^{\frac{5}{9}}_\gamma||U||^{\frac{4}{9}}_\gamma + |b|^{\frac{4}{9}}||Z||^{\frac{4}{9}}_\gamma||U||^{\frac{5}{9}}_\gamma$$
$$+ |b|^{\frac{1}{9}}||Z||^{\frac{1}{9}}_\gamma||U||^{\frac{8}{9}}_\gamma + ||U||_\gamma).$$

Step 3 Cont'ed

Young inequality then yields

$$|b||v_2|_2 \leq \varepsilon|b|||Z||_\gamma + C_\varepsilon||U||_\gamma, \quad \forall b \neq 0.$$

Step 4

This step is devoted to showing the estimate

$$b^2|\Delta u|_2^2 + \frac{b^2}{\gamma}\int_\Gamma |\partial_\nu u|^2\, d\Gamma \le \varepsilon^2 b^2 ||Z||_\gamma^2 + C_\varepsilon ||U||_\gamma^2.$$

For this purpose, set $u = u_1 + u_2$ with

$$ibu_1 = v_1 + f, \quad ibu_2 = v_2 = -\frac{\Delta^2 u + \alpha\Delta w + \Delta v_1}{ib}.$$

Notice that the right equation may be recast as

$$b^2\Delta^2 u = b^4 u_2 - \alpha b^2 \Delta w - b^2 \Delta v_1.$$

Step 4 Cont'ed

Multiplying this equation by $\bar{u}$ and using Green's formula, one gets

$$b^2|\Delta u|_2^2 + \frac{b^2}{\gamma}\int_\Gamma |\partial_\nu u|^2 \, d\Gamma = b^4|u_2|_2^2 + \Re\int_\Omega \{b^4 u_2\bar{u}_1 - b^2\alpha w\Delta\bar{u} - b^2 v_1\Delta\bar{u}\}\, dx.$$

Now, one checks

$$\begin{aligned} b^4\int_\Omega u_2\bar{u}_1\, dx &= \int_\Omega (ib\bar{v}_1 + ib\bar{f})(-ibv_2)\, dx \\ &= b^2\int_\Omega \bar{v}_1 v_2\, dx + ib\int_\Omega \bar{f}(\Delta^2 u + \alpha\Delta w + \Delta v_1)\, dx \\ &= b^2\int_\Omega \bar{v}_1 v_2\, dx + ib\int_\Omega \Delta u\Delta\bar{f}\, dx + \frac{ib}{\gamma}\int_\Gamma \partial_\nu u\partial_\nu\bar{f}\, d\Gamma \\ &\quad + ib\int_\Omega (\alpha w + v_1)\Delta\bar{f}\, dx. \end{aligned}$$

Step 4 Cont'ed

Thanks to Hölder and Young inequalities, one derives from those two equations:

$$b^2|\Delta u|_2^2 + \frac{b^2}{\gamma}\int_\Gamma |\partial_\nu u|^2\, d\Gamma \leq C_0 b^2(|v_1|_2^2 + |v_2|^2 + |w|_2^2) + C_0||U||_\gamma^2.$$

From Steps 2 and 3, it follows:

$$|b|||Z||_\gamma \leq \varepsilon|b|||Z||_\gamma + C_\varepsilon||U||_\gamma, \quad \forall\varepsilon > 0,\ \forall b \neq 0.$$

choosing $\varepsilon = 1/2$, and using Step 1, one gets:

$$(|b| + 1)||Z||_\gamma \leq C_0||U||_\gamma, \quad \forall\gamma > 0,\ \forall b \neq 0.$$

The case $b = 0$ is pretty straightforward.

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$$\begin{cases} y_{\gamma,tt} + \Delta^2 y_\gamma + \alpha\Delta\theta_\gamma = 0 \text{ in } \Omega\times(0,\infty) \\ \theta_{\gamma,t} - \kappa\Delta\theta_\gamma + \eta\theta_\gamma - \beta\Delta y_{\gamma,t} = 0 \text{ in } \Omega\times(0,\infty) \\ \gamma(\Delta y_\gamma + (1-\mu)B_1 y_\gamma + \alpha\theta_\gamma) + \partial_\nu y_\gamma = 0 \text{ on } \Sigma = \Gamma\times(0,\infty) \\ \gamma(\partial_\nu\Delta y_\gamma + (1-\mu)B_2 y - y_\gamma + \alpha\partial_\nu\theta) - y_\gamma = 0 \text{ on } \Sigma = \Gamma\times(0,\infty) \\ \partial_\nu\theta_\gamma + \lambda\theta_\gamma = 0 \text{ on } \Sigma \\ y_\gamma(0) = y_\gamma^0, \quad y_t(0) = y^1, \quad \theta_\gamma(0) = \theta^0 \text{ in } \Omega, \end{cases}$$

where $B_1 y = 2\nu_1\nu_2 y_{x_1x_2} - \nu_1^2 y_{x_2x_2} - \nu_2^2 y_{x_1x_1}$ and $B_2 y = \partial_\tau[(\nu_1^2 - \nu_2^2)y_{x_1x_2} + \nu_1\nu_2(y_{x_2x_2} - y_{x_1x_1})]$.

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Formally letting $\gamma\to\infty$ yields the free-boundary plate system whose semigroup analyticity was discussed by Lasiecka and Triggiani [Ann. Scu. Norm. Pisa, 1998].

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!