

# Some contributions to the simultaneous and indirect stabilization of multi-component systems

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# Overview

- An abstract model

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- Simultaneous stabilization

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  - Brief literature

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  - Lamé systems with localized damping

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- Indirect stabilization



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- An abstract model
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  - Timoshenko beam with localized damping
- Indirect stabilization
  - Brief literature

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- An abstract model
- Simultaneous stabilization
  - Brief literature
  - Lamé systems with localized damping
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  - Timoshenko beam with localized damping
- Indirect stabilization
  - Brief literature
  - Kirchhoff plate-wave

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- An abstract model
- Simultaneous stabilization
  - Brief literature
  - Lamé systems with localized damping
  - Euler-Bernoulli plate-wave system with localized damping
  - Timoshenko beam with localized damping
- Indirect stabilization
  - Brief literature
  - Kirchhoff plate-wave
  - Mindlin-Timoshenko plate

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- An abstract model
- Simultaneous stabilization
  - Brief literature
  - Lamé systems with localized damping
  - Euler-Bernoulli plate-wave system with localized damping
  - Timoshenko beam with localized damping
- Indirect stabilization
  - Brief literature
  - Kirchhoff plate-wave
  - Mindlin-Timoshenko plate
- Some extensions and open problems

## Model formulation

Let  $H$  and  $V$  be Hilbert spaces with  $V \subset H$ . Assume  $V$  is dense in  $H$  and the injection of  $V$  into  $H$  is compact. Denote by  $(\cdot, \cdot)$  the inner product in  $H$ , by  $|\cdot|$  the corresponding norm, and by  $V'$  the dual of  $V$ . Consider the damped abstract equation

$$\begin{aligned}y_{tt} + Ay + By_t &= 0 \text{ in } (0, \infty) \\ y(0) = y^0 \in V, \quad y_t(0) = y^1 \in H,\end{aligned}$$

where  $A \in \mathcal{L}(V, V')$  is a selfadjoint coercive operator with  $D(A^{\frac{1}{2}}) = V$ , and  $B \in \mathcal{L}(H)$  is a nonnegative operator.

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where  $A \in \mathcal{L}(V, V')$  is a selfadjoint coercive operator with  $D(A^{\frac{1}{2}}) = V$ , and  $B \in \mathcal{L}(H)$  is a nonnegative operator.

Introduce the energy

$$E(t) = \frac{1}{2} \{ |y_t(t)|^2 + |A^{\frac{1}{2}}y(t)|^2 \}, \quad \forall t \geq 0.$$

## Theorem: Dafermos criterion

1970: Dafermos proves: the abstract system is strongly stable

$$\lim_{t \rightarrow \infty} E(t) = 0$$

if and only if

$$\text{Ker} B \cap \text{Ker}(A + \lambda I) = \{0\}, \quad \forall \lambda \in \mathbb{R}$$

where  $I$  denotes the identity operator on  $H$ .

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where  $I$  denotes the identity operator on  $H$ .

For the stabilization of single component systems, we refer to the contributions of Rauch-Taylor, Bardos-Lebeau-Rauch, Russell, Dafermos, Chen, Haraux, Komornik, Lasiecka, Nakao, Liu, Martinez, Slemrod, Triggiani, Zuazua,...



## Brief literature

By simultaneous stabilization, we should understand stabilizing a multi-component system using the same damping mechanism in all components; the matrix defining the damping is degenerate.

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- 1986: Russell introduces the notion of simultaneous control for pdes when studying the boundary controllability of the Maxwell's equations.
- 1988: Lions (v.1, Controllability book) analyses simultaneous boundary control problems for two uncoupled waves, and for two uncoupled plates.

## Brief literature

Consider the system of uncoupled wave equations

$$u_{jtt} - a_j \Delta u_j = 0 \text{ in } Q$$

$$u_j = 0 \text{ on } \Gamma \times (0, T)$$

$$u_j(x, 0) = u_j^0(x), \quad u_{jt}(x, 0) = u_j^1(x) \text{ in } \Omega, \quad j = 1, 2, \dots, q,$$

where  $(u_j^0, u_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$  for each  $j$ .

1988: Haraux (1988) shows for arbitrary nonempty open set  $\omega$ :

- If  $\sum_{j=1}^q u_j(x, t) = 0$  in  $\omega \times (0, T)$  then  $u_j^0 = 0, \quad u_j^1 = 0$  in  $\Omega, \quad \forall j$ .  
provided that  $a_j \neq a_k$  for all  $j, k$  with  $j \neq k$ .

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provided that  $a_j \neq a_k$  for all  $j, k$  with  $j \neq k$ .
- If  $N = 1$  and  $T$  is large enough, or  $\omega = \Omega$ , then there exists  $C > 0$ :  
for all  $j$  and all  $(u_j^0, u_j^1) \in L^2(\Omega) \times H^{-1}(\Omega)$

$$\sum_{j=1}^q \{ \|u_j^0\|_{L^2(\Omega)}^2 + \|u_j^1\|_{H^{-1}(\Omega)}^2 \} \leq C \int_0^T \int_{\omega} \left| \sum_{j=1}^q u_j(x, t) \right|^2 dx dt$$

provided that  $a_j \neq a_k$  for all  $j, k$  with  $j \neq k$ .

**(GCC):** Bardos-Lebeau-Rauch (1992):  $\omega$  is an admissible control region in time  $T$  if every ray of geometric optics enters  $\omega$  in a time less than  $T$ .

### Theorem 1 (2012)

Let  $T_0$  denote the best controllability time for a single wave equation with unit speed of propagation. Suppose that

$T > T_0 \max\{a_j^{-\frac{1}{2}}; j = 1, 2, \dots, q\}$  and  $(\omega, T)$  satisfies (GCC). There exists a constant  $C > 0$  such that for all  $(u_j^0, u_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$ ,  $j = 1, 2, \dots, q$ :

$$\sum_{j=1}^q \{ \|u_j^0\|_{H_0^1(\Omega)}^2 + \|u_j^1\|_{L^2(\Omega)}^2 \} \leq C \int_0^T \int_{\omega} \left| \sum_{j=1}^q u_{jt}(x, t) \right|^2 dx dt,$$

with  $C = C(\Omega, \omega, T, (a_j)_j, q)$ .

## Lamé systems with localized damping

Given  $(y_j^0, y_j^1)_j \in \left( [H_0^1(\Omega)]^N \times [L^2(\Omega)]^N \right)^q$ , and a function  $d \in L^\infty(\Omega)$ ,  $d \geq 0$ , consider the damped elastodynamic system

$$y_{jtt} - \mu_j \Delta y_j - (\mu_j + \lambda_j) \nabla \operatorname{div}(y_j) + d \sum_{k=1}^q y_{kt} = 0 \text{ in } \Omega \times (0, \infty)$$

$$y_j = 0 \text{ on } \Gamma \times (0, \infty)$$

$$y_j(x, 0) = y_j^0(x), \quad y_{jt}(x, 0) = y_j^1(x), \text{ in } \Omega,$$

$$j = 1, 2, \dots, q,$$

where, for each  $j$ ,  $\mu_j$  and  $\lambda_j$  are the Lamé constants.

The total energy is given, for all  $t \geq 0$ , by

$$2E(t) = \sum_{j=1}^q \int_{\Omega} \{ |y_{jt}(x, t)|^2 + \mu_j |\nabla y_j(x, t)|^2 + (\mu_j + \lambda_j) |\operatorname{div}(y_j(x, t))|^2 \} dx$$



## Lamé systems with localized damping

$E$  is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = - \int_{\Omega} d(x) \left| \sum_{k=1}^q y_{kt}(x, t) \right|^2 dx.$$

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**Question 1:** Does the energy  $E$  decay to zero as time goes to infinity?

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**Question 1:** Does the energy  $E$  decay to zero as time goes to infinity?

**Question 2:** Under which conditions is the Lamé system exponentially stable?

# Lamé systems with localized damping

## Theorem 2: Strong stability

Let  $\omega$  be a nonempty subset of  $\Omega$ . Suppose that  $d$  is positive in  $\omega$ . The elastodynamic system is strongly stable:

$$\lim_{t \rightarrow \infty} E(t) = 0$$

if and only if the propagation speeds are pairwise distinct:

$$\mu_j \neq \mu_k \text{ and } \lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k, \quad \forall j, k, j \neq k.$$

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**Proof method:** Apply Dafermos criterion, or Benchimol or Arendt-Batty strong stability criterion, some linear algebra argument, and Imanuvilov-Yamamoto Carleman estimate.

# Lamé systems with localized damping

## Theorem 3: Exponential stability

Let  $(y_j^0, y_j^1)_j \in \left( [H_0^1(\Omega)]^N \times [L^2(\Omega)]^N \right)^q$ . Suppose

$$\mu_j \neq \mu_k, \quad \lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k, \quad \text{and} \quad \lambda_j \mu_k = \lambda_k \mu_j, \quad \forall j, k, j \neq k.$$

Assume that  $\omega$  satisfies the Liu geometric control condition, and suppose that the damping is effective in  $\omega$ :

$$\exists d_0 > 0 : d(x) \geq d_0 \text{ a.e. } \omega.$$

There exist positive constants  $M$  and  $\kappa$ , independent of the initial data, such that the following energy decay estimate holds:

$$E(t) \leq M e^{-\kappa t} E(0), \text{ for all } t \geq 0.$$

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$$E(t) \leq M e^{-\kappa t} E(0), \text{ for all } t \geq 0.$$

**Proof method:** FDM, multipliers technique, Huang or Prüss criterion.

## An observability result

Let  $T > 0$ . Let  $\omega$  be a nonempty open set in  $\Omega$  satisfying the Liu geometric control condition. Consider the uncoupled elastodynamic system

$$\begin{aligned}y_{jtt} - \mu_j \Delta y_j - (\mu_j + \lambda_j) \nabla \operatorname{div}(y_j) &= 0 \text{ in } \Omega \times (0, T) \\y_j &= 0 \text{ on } \Gamma \times (0, T) \\y_j(x, 0) &= y_j^0(x), \quad y_{jt}(x, 0) = y_j^1(x), \text{ in } \Omega, \\j &= 1, 2, \dots, q.\end{aligned}$$

There exists  $T_0 > 0$  such that for any  $T > T_0$ , there exists  $C > 0$ :

$$E(0) \leq \int_0^T \int_{\omega} \left| \sum_{j=1}^q y_{jt}(x, t) \right|^2 dx dt,$$

provided that

$$\mu_j \neq \mu_k, \quad \lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k, \quad \text{and} \quad \lambda_j \mu_k = \lambda_k \mu_j, \quad \forall j, k, j \neq k.$$



## Euler-Bernoulli Plate-wave system

Consider the damped system

$$\left\{ \begin{array}{l} y_{tt} - \Delta y + d(x)(y_t + z_t) = 0 \text{ in } \Omega \times (0, \infty) \\ z_{tt} + \Delta^2 z + d(x)(y_t + z_t) = 0 \text{ in } \Omega \times (0, \infty) \\ y = 0, \quad z = 0, \quad \partial_\nu z = 0 \text{ on } \Gamma \times (0, \infty) \\ y(0) = y^0 \in H_0^1(\Omega), \quad y_t(0) = y^1 \in L^2(\Omega), \\ z(0) = z^0 \in H_0^2(\Omega), \quad z_t(0) = z^1 \in L^2(\Omega). \end{array} \right.$$

The total energy is given, for all  $t \geq 0$ , by

$$2E(t) = \int_{\Omega} \{ |y_t(x, t)|^2 + |\nabla y(x, t)|^2 + |z_t(x, t)|^2 + |\Delta z(x, t)|^2 \} dx,$$

## Euler-Bernoulli Plate-wave system

$E$  is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = - \int_{\Omega} d(x) |y_t(x, t) + z_t(x, t)|^2 dx.$$

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**Question 1:** Does the energy  $E$  decay to zero as time goes to infinity?

**Question 2:** Under which conditions is the system exponentially stable?

# Euler-Bernoulli Plate-wave system

## Theorem 4: Strong stability

Let  $\omega$  be an arbitrary nonvoid open set contained in  $\Omega$ . Suppose that the damping coefficient  $d$  is positive in  $\omega$ .

The system is strongly stable:

$$\lim_{t \rightarrow \infty} E(t) = 0$$

provided that either  $\text{meas}(\partial\omega \cap \partial\Omega) > 0$ , or else, the only solution of  $\Delta u = -u$  in  $\Omega$  and  $u = 0$  on  $\partial\Omega$  is  $u = 0$ .

# Euler-Bernoulli Plate-wave system

## Theorem 5: Exponential stability

Let  $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$  and  $(z^0, z^1) \in H_0^2(\Omega) \times L^2(\Omega)$ .

Assume that  $\omega$  satisfies the Liu geometric control condition, and suppose that the damping is effective in  $\omega$ :

$$\exists d_0 > 0 : d(x) \geq d_0 \text{ a.e. } \omega.$$

There exist positive constants  $M$  and  $\kappa$ , independent of the initial data, such that the following energy decay estimate holds:

$$E(t) \leq Me^{-\kappa t} E(0), \text{ for all } t \geq 0.$$

## An observability inequality

Let  $T > 0$ . Let  $\omega$  be a nonempty open set in  $\Omega$  satisfying the Liu geometric control condition.

Consider the uncoupled system

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } \Omega \times (0, T) \\ z_{tt} + \Delta^2 z = 0 & \text{in } \Omega \times (0, T) \\ y = 0, \quad z = 0, \quad \partial_\nu z = 0 & \text{on } \Gamma \times (0, T). \end{cases}$$

There exists  $T_0 > 0$  such that for any  $T > T_0$ , there exists  $C > 0$ :

$$E(0) \leq \int_0^T \int_\omega |y_t(x, t) + z_t(x, t)|^2 dx dt,$$

## Timoshenko beam

Let  $L > 0$ , and set  $\Omega = (0, L)$ , and  $\omega = (l_1, l_2)$  with  $0 \leq l_1 < l_2 \leq L$ . Consider the damped Timoshenko system:

$$\begin{cases} \rho_1 y_{tt} - k(y_x + z)_x + a(x)(y_t + z_t) = 0 & \text{in } (0, L) \times (0, \infty) \\ \rho_2 z_{tt} - \sigma z_{xx} + k(y_x + z) + a(x)(y_t + z_t) = 0 & \text{in } (0, L) \times (0, \infty), \end{cases}$$

with the boundary conditions:

**(DD)**  $y(0, t) = 0, \quad y(L, t) = 0, \quad z(0, t) = 0, \quad z(L, t) = 0$ , or else

**(DN)**  $y(0, t) = 0, \quad y(L, t) = 0, \quad z_x(0, t) = 0, \quad z_x(L, t) = 0, \quad t > 0$

and the initial conditions:

$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x),$



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$$\begin{cases} \rho_1 y_{tt} - k(y_x + z)_x + a(x)(y_t + z_t) = 0 & \text{in } (0, L) \times (0, \infty) \\ \rho_2 z_{tt} - \sigma z_{xx} + k(y_x + z) + a(x)(y_t + z_t) = 0 & \text{in } (0, L) \times (0, \infty), \end{cases}$$

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$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x),$

The damping coefficient  $a$  is a nonnegative bounded measurable function, which is positive in  $\omega$  only.

# The energy and main questions

Introduce the energy

$$E(t) = \frac{1}{2} \int_{\Omega} \{ \rho_1 |y_t(x, t)|^2 + k |y_x(x, t) + z(x, t)|^2 \} dx \\ + \frac{1}{2} \int_{\Omega} \{ \rho_2 |z_t(x, t)|^2 + \sigma |z_x(x, t)|^2 \} dx, \quad \forall t \geq 0.$$

The energy  $E$  is a nonincreasing function of the time variable  $t$  as we have for every  $t \geq 0$ , (hereafter,  $'$  denotes differentiation with respect to time)

$$E'(t) = - \int_{\Omega} a(x) |y_t(x, t) + z_t(x, t)|^2 dx.$$

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$$E'(t) = - \int_{\Omega} a(x) |y_t(x, t) + z_t(x, t)|^2 dx.$$

As before, our main purpose is to answer the following questions:

- Does the energy  $E(t)$  decay to zero as the time variable  $t$  goes to infinity?
- If so, how fast? And if not, why?

# Timoshenko beam

## Theorem 6: Strong stability

Suppose that  $\omega$  is an arbitrary nonempty open interval in  $\Omega$ . Let the damping coefficient  $a$  be positive in  $\omega$ . In either of the **(DD)** or **(DN)** case, the associated system is strongly stable:

$$\lim_{t \rightarrow \infty} E(t) = 0$$

if and only if  $\partial\omega \cap \partial\Omega \neq \emptyset$ .

# Timoshenko beam

## Theorem 7: Exponential stability

Suppose that  $\omega$  is an arbitrary nonempty open interval in  $\Omega$  with  $\partial\omega \cap \partial\Omega \neq \emptyset$ . Let the damping coefficient  $a$  satisfy

$$a(x) \geq a_0 > 0, \text{ a.e. in } \omega.$$

There exist positive constants  $M$  and  $\kappa$ , independent of the initial data, such that the following energy decay estimate holds:

$$E(t) \leq Me^{-\kappa t} E(0), \text{ for all } t \geq 0.$$

## Brief literature

Notion explicitly introduced by Russell (1993), involves a coupled system of second order evolution equations where the damping occurs in one component of the system only.

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We can broaden the notion to account for thermoelasticity or fluid-structure models where the dissipation is induced by the heat or parabolic component only.

Other contributors include Dafermos, Lasiecka and collaborators, Burns and collaborators, Lebeau-Zuazua, Perla Menzala-Zuazua, Rauch-Zhang-Zuazua, Triggiani-Avalos, Zhang-Zuazua, Alabau, Alabau-Cannarsa-Komornik,...

## Mindlin-Timoshenko plate

$$\rho_1 y_{tt} - k \operatorname{div}(\nabla y + z) = 0 \text{ in } \Omega \times (0, \infty)$$

$$\rho_2 z_{tt} - \mu \Delta z - (\lambda + \mu) \nabla \operatorname{div} z + k(\nabla y + z) + a z_t = 0 \text{ in } \Omega \times (0, \infty)$$

$$y = 0, \quad z = 0 \text{ on } \partial\Omega \times (0, \infty)$$

$$y(\cdot, 0) = y^0 \in H_0^1(\Omega), \quad y_t(\cdot, 0) = y^1 \in L^2(\Omega),$$

$$z(\cdot, 0) = z^0 \in [H_0^1(\Omega)]^N, \quad z_t(\cdot, 0) = z^1 \in [L^2(\Omega)]^N.$$



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In the one-dimensional setting, the system, known as the Timoshenko beam equations, describes the motion of a beam when the effects of rotatory inertia are accounted for; the transverse displacement is represented by  $y$  while  $z$  denotes the shear angle displacement.

In 2D, that system is known as the Mindlin-Timoshenko plate equations, where  $y$  represents the vertical deflection and  $z$  stands for the rotation angles of a filament.

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The constants  $\rho_1$ ,  $\rho_2$ ,  $k$ , and  $\mu$  are physical constants and are all positive. In particular, the constants  $\lambda$  and  $\mu$  are the Lamé constants with  $\lambda + \mu > 0$ .

## Mindlin-Timoshenko plate

2009: Fernández-Sare shows that the system is polynomially stable.

# Mindlin-Timoshenko plate

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$$(*) \quad \frac{k}{\rho_1} = \frac{2\mu + \lambda}{\rho_2}.$$

**Questions:** Is the Mindlin-Timoshenko system exponentially stable under (\*)? **What happens when (\*) fails?**



## Energy estimates

Introduce the energy

$$E(t) = \frac{1}{2} \int_{\Omega} \{ \rho_1 |y_t(x, t)|^2 + k |\nabla y(x, t) + z(x, t)|^2 \} dx \\ + \frac{1}{2} \int_{\Omega} \{ \rho_2 |z_t(x, t)|^2 + \mu |\nabla z(x, t)|^2 + (\lambda + \mu) |\operatorname{div} z(x, t)|^2 \} dx, \quad \forall t \geq 0.$$

Let  $\omega$  satisfy Liu geometric constraint. Suppose that the damping coefficient  $a$  further satisfies

$$\exists a_0 > 0 : a(x) \geq a_0, \text{ a.e. } \omega.$$

## Energy estimates

- If (\*) holds, then the energy decays exponentially:

$$\exists M > 0, \exists \zeta > 0 : E(t) \leq M e^{-\zeta t} E(0), \quad \forall t \geq 0.$$

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$$\exists M > 0, \exists \zeta > 0 : E(t) \leq M e^{-\zeta t} E(0), \quad \forall t \geq 0.$$

- If (\*) fails, then the energy decays polynomially:

$$\exists M = M(\text{initial data}) > 0, \exists \zeta > 0 : E(t) \leq \frac{M}{(1+t)},$$

provided

$$(y^0, y^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$$

and

$$(z^0, z^1) \in [(H^2(\Omega) \cap H_0^1(\Omega))]^N \times [H_0^1(\Omega)]^N.$$

# Kirchhoff plate-wave-2019

Joint work with Ahmed Hajej (U. Cergy-Pontoise, France) and Zayd Hajjej (U. Gabes, Tunisia)

## Undamped Kirchhoff plate/ damped wave

Consider the following weakly coupled system of Kirchhoff plate and wave equations:

$$\left\{ \begin{array}{ll} u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + \alpha v = 0 & \text{in } \Omega \times (0, \infty) \\ v_{tt} - \Delta v + v_t + \alpha u = 0 & \text{in } \Omega \times (0, \infty) \\ u = \partial_\nu u = 0 & \text{on } \Gamma_0 \times (0, \infty) \\ \Delta u + (1 - \mu) B_1 u = 0 & \text{on } \Gamma_1 \times (0, \infty) \\ \partial_\nu \Delta u - \gamma \partial_\nu u_{tt} + (1 - \mu) B_2 u = 0 & \text{on } \Gamma_1 \times (0, \infty) \\ v = 0 & \text{on } \Gamma \times (0, \infty) \\ u(0) = u^0 \in V, \quad u_t(0) = u^1 \in H_0^1(\Omega), \\ v(0) = v^0 \in H_0^1(\Omega), \quad v_t(0) = v^1 \in L^2(\Omega). \end{array} \right.$$

## Undamped Kirchhoff plate/ damped wave

$\Omega$  is an open set of  $\mathbb{R}^2$  with regular boundary  $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$  such that  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ ,

The constant  $\gamma > 0$  is the rotational inertia of the plate and the constant  $0 < \mu < \frac{1}{2}$  is the Poisson coefficient.

The boundary operators  $B_1, B_2$  are defined by

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx},$$

$$B_2 u = \partial_\tau \left( (\nu_1^2 - \nu_2^2) u_{xy} + \nu_1\nu_2 (u_{yy} - u_{xx}) \right),$$

where  $\nu = (\nu_1, \nu_2)$  is the unit outer normal vector to  $\Gamma$  and  $\tau = (-\nu_2, \nu_1)$  is a unit tangent vector.

## Energy estimates.

Introduce the energy, (setting  $P_\gamma u = u - \gamma \Delta u$ )

$$E(t) = \frac{1}{2} \int_{\Omega} \{ |P_\gamma^{\frac{1}{2}} u_t|^2 + |\Delta u|^2 + |v_t|^2 + |\nabla v|^2 + 2\alpha uv \} (x, t) dx.$$

We have:

$$E(t) \leq \frac{C_\alpha}{(t+1)^{\frac{1}{3}}} \left( \|u^0\|_{H^3(\Omega)}^2 + \|u^1\|_{H^2(\Omega)}^2 + \|v^0\|_{H^2(\Omega)}^2 + \|v^1\|_{H_0^1(\Omega)}^2 \right).$$

FDM, interpolation, good choice of functional inequalities,  
Borichev-Tomilov criterion.

## Damped Kirchhoff plate/ undamped wave

$$\left\{ \begin{array}{ll} u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + \alpha v + u_t = 0 & \text{in } \Omega \times (0, \infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in } \Omega \times (0, \infty) \\ u = \partial_\nu u = 0 & \text{on } \Gamma_0 \times (0, \infty) \\ \Delta u + (1 - \mu) B_1 u = 0 & \text{on } \Gamma_1 \times (0, \infty) \\ \partial_\nu \Delta u - \gamma \partial_\nu u_{tt} + (1 - \mu) B_2 u = 0 & \text{on } \Gamma_1 \times (0, \infty) \\ v = 0 & \text{on } \Gamma \times (0, \infty) \\ u(0) = u^0 \in V, \quad u_t(0) = u^1 \in H_0^1(\Omega), \\ v(0) = v^0 \in H_0^1(\Omega), \quad v_t(0) = v^1 \in L^2(\Omega). \end{array} \right.$$



## Energy estimates.

Introduce the energy

$$E(t) = \frac{1}{2} \int_{\Omega} \{ |p_{\gamma} u_t|^2 + |\Delta u|^2 + |v_t|^2 + |\nabla v|^2 + 2\alpha uv \} (x, t) dx.$$

We have:

$$E(t) \leq \frac{C_{\alpha}}{(t+1)^{\frac{1}{4}}} \left( \|u^0\|_{H^3(\Omega)}^2 + \|u^1\|_{H^2(\Omega)}^2 + \|v^0\|_{H^2(\Omega)}^2 + \|v^1\|_{H_0^1(\Omega)}^2 \right).$$

- ① Obtaining logarithmic energy decay estimates in the case of simultaneous stabilization in the multidimensional setting when  $\omega$  is an arbitrary nonempty open subset of  $\Omega$ .

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And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!