Precise decay estimates for semigroups associated with some one-dimensional fluid-structure interactions involving degeneracy

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Overview

- A brief history

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- Wave/parabolic model

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- Euler-Bernoulli/parabolic model

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- Euler-Bernoulli/parabolic model
- Final comments and open problems

Initial work

In 2003, Zhang and Zuazua considered the following system

$$\begin{cases} y_{tt} - y_{xx} = 0 \text{ in } (-1,0) \times (0,\infty) \\ z_t - z_{xx} = 0 \text{ in } (0,1) \times (0,\infty) \\ y(0) = y^0 \in V, \quad y_t(0) = y^1 \in L^2(-1,0) \\ z(0) = z^0 \in L^2(0,1) \\ y(-1,t) = 0, \quad z(1,t) = 0 \text{ in } (0,\infty) \\ y_t(0-,t) = z(0+,t), \quad y_x(0-,t) = z_x(0+,t) \text{ in } (0,\infty). \end{cases}$$

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Energy Dissipation

The energy of this system is given by:

$$E(t) = \frac{1}{2}\int_{-1}^{0} \{|y_t(x,t)|^2 + |y_x(x,t)|^2\}\,dx + \int_0^1 |z(x,t)|^2\,dx$$

and it is nonincreasing, as we have the dissipation law:

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They proved the optimal decay of the energy (Riesz basis method):

$$\exists C_0 > 0 : E(t) \leq \frac{C_0\left(||y^0||^2_{H^2(-1,0)} + ||y^1||^2_V + ||z^0||^2_{H^1(0,1)}\right)}{(1+t)^4}, \quad \forall t \geq 0.$$

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They conjectured that the exponent 1/3 should be replaced by 2.

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$\forall s < 2,\ \exists C_s > 0:$

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The geometric optics approach is utilized and the domain must have a C^∞ boundary.

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$$\exists C_0 > 0, \exists \alpha > 0 : E(t) \leq C_0 e^{-\alpha t} E(0), \quad \forall t \geq 0.$$

A nonlinear counterpart of this Avalos-Triggiani work (linear wave/Navier-Stokes equations) was analyzed by Lasiecka and Lu [2012].

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A new system

Let $\alpha \in (0,1)$ be a constant. Let $a \in C^1([0,1])$. Consider the following hyperbolic/parabolic transmission system

$$
\begin{cases}
y_{tt} - (a(x) y_x)_x = 0 \text{ in } (-1,0) \times (0,\infty) \\
z_t - (x^\alpha z_x)_x = 0 \text{ in } (0,1) \times (0,\infty) \\
y(-1,t) = 0, \quad z(1,t) = 0 \text{ in } (0,\infty) \\
y_t(0-,t) = z(0+,t), \quad a(0) y_x(0-,t) = (x^\alpha z_x)(0+,t) \text{ in } (0,\infty). \\
y(0) = y^0 \in V,\ y_t(0) = y^1 \in L^2(-1,0),\ z(0) = z^0 \in L^2(0,1),
\end{cases}
$$

and

$$\exists a_0 > 0 : a(x) \geq a_0, \quad \forall x \in [-1,0].$$

The energy of the new system

The energy of this system is given by:

$$E(t) = \frac{1}{2} \int_{-1}^{0} \{ |y_t(x,t)|^2 + a(x) |y_x(x,t)|^2 \} \, dx + \int_0^1 |z(x,t)|^2 \, dx.$$

Now, we have the dissipation law:

$$\frac{dE}{dt}(t) = - \int_0^1 x^\alpha |z_x(x,t)|^2 \, dx, \quad \forall t \geq 0.$$

A new decay rate

This new system with $a \equiv 1$ was first considered by Han, Wang and Wang [2020] who showed that the corresponding semigroup $(S_\alpha(t))_{t\geq 0}$ decays at the rate $O(t^{-\frac{(3-\alpha)}{2(1-\alpha)}})$.

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- Positive: The decay rate depends on α and becomes better and better as α approaches 1.
- Negative: The decay rate is clearly not optimal; as $\alpha \searrow 0$, the rate is just $O(t^{-\frac{3}{2}})$, which is very far from the optimal decay rate $O(t^{-2})$.

Theorem 1 (2022)

(Wave/parabolic model) For every $\alpha \in (0,1)$, there exist positive constants K_0 and K_α such that the semigroup $(S_\alpha(t))_{t\geq 0}$ satisfies for every $t \geq 0$:

$$||S_\alpha(t)Z^0|| \leq \begin{cases} \dfrac{K_0||Z^0||_{D(\mathcal{A}_\alpha)}}{(1+t)^2} \text{ if } 0 < \alpha \leq 1/4 \\[2ex] \dfrac{K_0||Z^0||_{D(\mathcal{A}_\alpha)}}{(1+t)^{\frac{3}{2(1-\alpha)}}}, \text{ if } 1/4 \leq \alpha \leq 1/2 \\[2ex] \dfrac{K_0||Z^0||_{D(\mathcal{A}_\alpha)}}{(1+t)^{\frac{(9-6\alpha)}{4(1-\alpha)}}}, \text{ if } 1/2 \leq \alpha \leq 3/4 \\[2ex] \dfrac{K_\alpha||Z^0||_{D(\mathcal{A}_\alpha)}}{(1+t)^{\frac{(3-\alpha)}{2(1-\alpha)}}}, \text{ if } 3/4 \leq \alpha < 1 \end{cases} \qquad \forall Z^0 \in D(\mathcal{A}_\alpha),$$

Key elements of the proof

Thanks to Borichev-Tomilov polynomial stability criterion [2010], it suffices to prove

- $i\mathbb{R} \subset \rho(\mathcal{A}_\alpha)$.
- There exists $C_\alpha > 0$ such that for every $U \in \mathcal{H}$, one has:

  $$||(i\lambda I - \mathcal{A}_\alpha)^{-1}U|| \leq C_\alpha |\lambda|^s ||U||, \quad \forall\, \lambda \in \mathbb{R}, \text{ with } |\lambda| \geq \lambda_\alpha$$

  for some $\lambda_\alpha > 1$,

with the exponent s given by

$$s = \begin{cases} 1/2 \text{ if } 0 < \alpha \leq 1/4, \\ \frac{2(1-\alpha)}{3} \text{ if } 1/4 \leq \alpha \leq 1/2, \\ \frac{4(1-\alpha)}{9-6\alpha} \text{ if } 1/2 \leq \alpha \leq 3/4, \\ \frac{2(1-\alpha)}{3-\alpha} \text{ if } 3/4 \leq \alpha < 1. \end{cases}$$

The transmission system

Let α in $[0,1)$. Let $d \in C^2([-1,0]$. Consider now the following system

$$\begin{cases} y_{tt} + (d(x)y_{xx})_{xx} = 0 \text{ in } (-1,0)\times(0,\infty) \\ z_t - (x^\alpha z_x)_x = 0 \text{ in } (0,1)\times(0,\infty) \\ y(-1,t) = 0, \quad y_x(-1,t) = 0, \quad y_{xx}(0-,t) = 0, \quad z(1,t) = 0 \text{ in } (0,\infty \\ y_t(0-,t) = z(0+,t), \quad (dy_{xx})_x(0-,t) = -(x^\alpha z_x)(0+,t) \text{ in } (0,\infty) \\ y(0) = y^0 \in W, \quad y_t(0) = y^1 \in L^2(-1,0) \\ z(0) = z^0 \in L^2(0,1), \end{cases}$$

with the space W given by

$$W = \{u \in H^2(-1,0); u(-1) = u_x(-1) = 0\},$$

and

$$\exists d_0 > 0 : d(x) \geq d_0, \; \forall x \in (-1,0).$$

The energy is dissipative

The energy of this system is given by

$$E_2(t) = \frac{1}{2}\int_{-1}^{0} \{|y_t(x,t)|^2 + d(x)|y_{xx}(x,t)|^2\}\, dx + \frac{1}{2}\int_0^1 |z(x,t)|^2\, dx,$$

and one readily checks that this energy is a noincreasing function of the time variable as

$$\frac{d}{dt}E_2(t) = -\int_0^1 x^\alpha |z_x(x,t)|^2\, dx, \text{ for a.e. } t \geq 0.$$

Theorem 2 (2022)

(**EB beam/parabolic model.**) For every $\alpha \in [0,1)$, the semigroup $(\widetilde{S}_\alpha(t))_{t\geq 0}$ is exponentially stable; there exist positive constants $K_0 \geq 1$, independent of α, and μ_α such that for every $t \geq 0$:

$$||\widetilde{S}_\alpha(t)Z^0|| \leq K_0 e^{-\mu_\alpha t}||Z^0||, \quad \forall Z^0 \in \widetilde{\mathcal{H}},$$

where the constant $\mu_\alpha \searrow 0$ as $\alpha \nearrow 1$.

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Work in progress: new results

Theorem 3

(**Wave/parabolic model.**) For every $\alpha \in (0,1)$, the semigroup $(S_\alpha(t))_{t\geq 0}$ is polynomially stable; there exists a positive constant K_α such that for every $t \geq 0$:

$$||S_\alpha(t)Z^0|| \leq \frac{K_\alpha ||Z^0||_{D(\mathcal{A}_\alpha)}}{(1+t)^{\frac{(2-\alpha)}{(1-\alpha)}}}, \quad \forall Z^0 \in D(\mathcal{A}_\alpha),$$

where the constant $K_\alpha \nearrow \infty$ as $\alpha \nearrow 1$.

Work in progress: new results

Theorem 4

(**EB beam/parabolic model.**) For every $\alpha \in [0,1)$, the semigroup $(\widehat{S}_\alpha(t))_{t\geq 0}$ is of Gevrey class δ for every $t > 0$, and for every $\delta > (2-\alpha)/(1-\alpha)$. In particular, there exists a positive constant C_α such that the following resolvent estimate holds

$$\limsup_{|\lambda|\to\infty} |\lambda|^{\frac{1-\alpha}{2-\alpha}} \|(i\lambda I - \widehat{\mathcal{A}}_\alpha)^{-1}\|_{\mathcal{L}(\widehat{\mathcal{H}})} \leq C_\alpha,$$

with $C_\alpha \nearrow \infty$ as $\alpha \nearrow 1$.

Open problems

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2. What happens when the degeneracy occurs inside the parabolic component domain instead of occuring at the interface?
3. Semigroup regularity issues for plate/parabolic models. For plate/parabolic models, exponential decay of the energy has been established, e.g. Avalos-Geredeli recent result [2020].

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!