Simultaneous controllability of some uncoupled semilinear wave equations

Louis Tebou

Florida International University

AMS Meeting
Knoxville

March 22, 2014

Overview

- The asymptotically linear case.

Overview

- The asymptotically linear case.
- The superlinear case.

Overview

- The asymptotically linear case.
- The superlinear case.
- Comments and open problems.

Notations

Ω= bounded domain in $\mathbb{R}^N$, $N \geq 1$,
Γ= boundary of Ω is smooth,
$T > 0$, $Q = \Omega \times (0, T)$
$\omega =$ nonvoid open subset in Ω.
$a_1,\ a_2, ...,\ a_q$, ($q \geq 2$) are pairwise distinct positive constants.
$\Delta_b = \partial_i(b_{ij}(x)\partial_j)$, (Einstein summation convention used on Latin letters, but not on Greek letters.)
The coefficients b_{ij} are smooth and satisfy the standard ellipticity condition.

Problem formulation

Consider the controllability problem: Given $(y_\alpha^0, y_\alpha^1)_\alpha$ and $(z_\alpha^0, z_\alpha^1)_\alpha$ in $\left(H_0^1(\Omega) \times L^2(\Omega)\right)^q$, and a smooth function $f : \mathbb{R} \to \mathbb{R}$ with

$$\exists C > 0,\ \sigma \geq 0 : |f(s) - f(r)| \leq C|s - r|(1 + |s|^\sigma + |r|^\sigma),\ \forall s,\ r \in \mathbb{R},$$
$$\limsup_{|s|\to\infty} \frac{f(s)}{s} = \lambda \text{ for some } \lambda \in \mathbb{R}$$
$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0 : |f(s) - \lambda s| \leq C_\varepsilon + \varepsilon|s|,\ \forall s \in \mathbb{R},$$

find a control $v \in L^2(0, T; L^2(\omega))$ such that if the q-tuple $(y_\alpha)_\alpha$ solves the system

$$\begin{cases} y_{\alpha,tt} - a_\alpha \Delta_b y_\alpha + a_\alpha f(y_\alpha) = v 1_\omega \text{ in } Q \\ y_\alpha = 0 \text{ on } \Gamma \times (0, T) \\ y_\alpha(x, 0) = y_\alpha^0(x), \quad y_{\alpha,t}(x, 0) = y_\alpha^1(x) \text{ in } \Omega, \\ \alpha = 1,\ 2, ...,\ q, \end{cases}$$

then for each $\alpha = 1,\ 2, ...,\ q$

$$y_\alpha(x, T) = z_\alpha^0(x), \quad y_{\alpha,t}(x, T) = z_\alpha^1(x),\ \text{in } \Omega.$$

Remark

- T and ω must be large enough.

Remark

- T and ω must be large enough.
- In particular, ω and T satisfy the geometric control condition: **(GCC)** [Bardos-Lebeau-Rauch, 1988, 1992]: every ray of geometric optics enters ω in a time less than T.

Remark

- T and ω must be large enough.
- In particular, ω and T satisfy the geometric control condition: **(GCC)** [Bardos-Lebeau-Rauch, 1988, 1992]: every ray of geometric optics enters ω in a time less than T.
- To tackle the nonlinear control problem at hand, we shall follow the standard procedure:

1. introduce a linearized control problem,

Remark

- T and ω must be large enough.
- In particular, ω and T satisfy the geometric control condition: **(GCC)** [Bardos-Lebeau-Rauch, 1988, 1992]: every ray of geometric optics enters ω in a time less than T.
- To tackle the nonlinear control problem at hand, we shall follow the standard procedure:

1. introduce a linearized control problem,
2. solve the linear control problem,

Remark

- T and ω must be large enough.
- In particular, ω and T satisfy the geometric control condition: **(GCC)** [Bardos-Lebeau-Rauch, 1988, 1992]: every ray of geometric optics enters ω in a time less than T.
- To tackle the nonlinear control problem at hand, we shall follow the standard procedure:

1. introduce a linearized control problem,
2. solve the linear control problem,
3. use a fixed-point theorem to derive the controllability of the nonlinear problem from that of the linearized system.

Linearized Control Problem

Let the data be as above. For each $1 \leq \alpha \leq q$, let $p_\alpha \in L^2(Q)$ be arbitrary. Introduce the linear controllability problem: Find a control $v \in L^2(0, T; L^2(\omega))$ such that if the q-tuple $(y_\alpha)_\alpha$ solves the system

$$\begin{cases} y_{\alpha,tt} - a_\alpha \Delta_b y_\alpha + a_\alpha \lambda y_\alpha = a_\alpha(-f(p_\alpha) + \lambda p_\alpha) + v1_\omega \text{ in } Q \\ y_\alpha = 0 \text{ on } \Gamma \times (0, T) \\ y_\alpha(x, 0) = y_\alpha^0(x), \quad y_{\alpha,t}(x, 0) = y_\alpha^1(x) \text{ in } \Omega, \\ \alpha = 1,\ 2, ...,\ q, \end{cases}$$

then for each $\alpha = 1,\ 2, ...,\ q$

$$y_\alpha(x, T) = z_\alpha^0(x), \quad y_{\alpha,t}(x, T) = z_\alpha^1(x) \text{ in } \Omega.$$

Section main results

Theorem 1

Suppose that Ω is of class C^3, and the coefficients b_{ij} lie in $C^2(\bar{\Omega})$. Assume that ω satisfies the Bardos-Lebeau-Rauch geometric control condition. Suppose that the constants a_α, $\alpha = 1,\ 2,\ ...,\ q$, are pairwise distinct. Assume that $\lambda > -\lambda_1$, where λ_1 is the first eigenvalue of $-\Delta_b$ with Dirichlet boundary conditions.

Let $T > T_0 \max\{a_\alpha^{-\frac{1}{2}}; \alpha = 1,\ 2,\ ...,\ q\}$, where T_0 is the controllability time corresponding to the single linear hyperbolic equation, ($f \equiv 0$), with $a_\alpha = 1$. For all (y_α^0, y_α^1), and (z_α^0, z_α^1) in $H_0^1(\Omega) \times L^2(\Omega)$, ($\alpha = 1,\ 2,\ ...,\ q$), there exists a control $v \in L^2(0, T; L^2(\omega))$ such that the corresponding solution of the linearized system satisfies the prescribed final conditions.

Proof of Theorem 1: key elements

First, we show the following observability estimate

$$\widehat{E}(0) \leq C_0 \int_0^T \int_\omega \left| \sum_{\alpha=1}^{q} u_\alpha(x,t) \right|^2 dxdt, \forall (u_\alpha^0, u_\alpha^1)_\alpha \in \left(L^2(\Omega) \times H^{-1}(\Omega) \right)^q,$$

where $2\widehat{E}(0) = \sum_{\alpha=1}^{q} \left(||u_\alpha^0||^2_{L^2(\Omega)} + ||u_\alpha^1||^2_{H^{-1}(\Omega)} \right)$, for every solution of the system

$$\begin{cases} u_{\alpha,tt} - a_\alpha \Delta_b u_\alpha + a_\alpha \lambda u_\alpha = 0 \text{ in } Q \\ u_\alpha = 0 \text{ on } \Sigma = \Gamma \times (0,T) \\ u_\alpha(.,0) = u_\alpha^0 \in L^2(\Omega), \quad u_{\alpha,t}(.,0) = u_\alpha^1 \in H^{-1}(\Omega), \quad \alpha = 1,\ 2, ...,\ q. \end{cases}$$

To prove the observability estimate, introduce for each α, the new function $w_\alpha(x,t) = \int_0^t u_\alpha(x,s)\,ds + z_\alpha(x)$, with $z_\alpha \in H_0^1(\Omega)$ satisfying $-a_\alpha \Delta_b z_\alpha + a_\alpha \lambda z_\alpha = -u_\alpha^1$ in Ω. Then w_α solves

$$\begin{cases} w_{\alpha,tt} - a_\alpha \Delta_b w_\alpha + a_\alpha \lambda w_\alpha = 0 \text{ in } Q \\ w_\alpha = 0 \text{ on } \Sigma = \Gamma \times (0,T) \\ w_\alpha(.,0) = z_\alpha \in H_0^1(\Omega), \quad w_{\alpha,t}(.,0) = u_\alpha^0 \in L^2(\Omega), \quad \alpha = 1,\ 2, ...,\ q. \end{cases}$$

To prove the observability estimate, introduce for each α, the new function $w_\alpha(x,t) = \int_0^t u_\alpha(x,s)\,ds + z_\alpha(x)$, with $z_\alpha \in H_0^1(\Omega)$ satisfying $-a_\alpha \Delta_b z_\alpha + a_\alpha \lambda z_\alpha = -u_\alpha^1$ in Ω. Then w_α solves

$$\begin{cases} w_{\alpha,tt} - a_\alpha \Delta_b w_\alpha + a_\alpha \lambda w_\alpha = 0 \text{ in } Q \\ w_\alpha = 0 \text{ on } \Sigma = \Gamma \times (0,T) \\ w_\alpha(.,0) = z_\alpha \in H_0^1(\Omega), \quad w_{\alpha,t}(.,0) = u_\alpha^0 \in L^2(\Omega), \quad \alpha = 1,\ 2, ...,\ q. \end{cases}$$

Thanks to Bardos-Lebeau-Rauch observability estimate for a single wave equation, one has for appropriate cut-off functions:

$$||z_\alpha||^2_{H_0^1(\Omega)} + ||u_\alpha^0||^2_{L^2(\Omega)} \leq C \int_0^T r(t) \int_\omega \eta |w_{\alpha,t}(x,t)|^2\,dxdt.$$

Elementary algebra shows

- $$\sum_{\alpha=1}^{q}\{||z_\alpha||^2_{H_0^1(\Omega)} + ||u_\alpha^0||^2_{L^2(\Omega)}\} \leq C\int_0^T\int_\omega |\sum_{\alpha=1}^{q} w_{\alpha,t}(x,t)|^2\,dxdt$$
$$-2C\sum_{1\leq\alpha<\beta\leq q}\int_Q r\eta w_{\alpha,t}w_{\beta,t}\,dxdt.$$

Elementary algebra shows

- $$\sum_{\alpha=1}^{q}\{||z_\alpha||^2_{H_0^1(\Omega)} + ||u^0_\alpha||^2_{L^2(\Omega)}\} \leq C\int_0^T\int_\omega |\sum_{\alpha=1}^{q} w_{\alpha,t}(x,t)|^2\,dxdt \\ -2C\sum_{1\leq\alpha<\beta\leq q}\int_Q r\eta w_{\alpha,t}w_{\beta,t}\,dxdt.$$
- Use Green's theorem and algebra to get:

$$2E(0) \leq C\int_0^T\int_\omega \left|\sum_{\alpha=1}^{q} w_{\alpha,t}(x,t)\right|^2 dxdt \\ -2C\sum_{1\leq\alpha<\beta\leq q}\frac{1}{a_\alpha - a_\beta}\left[\int_Q r'\eta(a_\beta w_\beta w_{\alpha,t} - a_\alpha w_\alpha w_{\beta,t})\,dxdt \right.\\ \left. +a_\alpha a_\beta\int_Q r(w_\alpha b_{ij}\partial_j w_\beta - w_\beta b_{ij}\partial_j w_\alpha)\partial_i\eta\,dxdt\right].$$

- With $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$, a combination of algebra and calculus shows

$$\sum_{\alpha=1}^{q} \{||z_\alpha||^2_{H_0^1(\Omega)} + ||u_\alpha^0||^2_{L^2(\Omega)}\}$$

$$\leq C \int_0^T \int_\omega |\sum_{\alpha=1}^{q} w_{\alpha,t}(x,t)|^2 \, dxdt$$
$$+C \int_Q \sum_{\alpha=1}^{q} |w_\alpha(x,t)|^2 \, dxdt.$$

- With $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$, a combination of algebra and calculus shows

$$\sum_{\alpha=1}^{q}\{||z_\alpha||^2_{H_0^1(\Omega)} + ||u^0_\alpha||^2_{L^2(\Omega)}\}$$

$$\leq C\int_0^T\int_\omega |\sum_{\alpha=1}^{q} w_{\alpha,t}(x,t)|^2\,dxdt$$
$$+C\int_Q \sum_{\alpha=1}^{q} |w_\alpha(x,t)|^2\,dxdt.$$

- Claim:

$$\int_Q \sum_{\alpha=1}^{q} |w_\alpha(x,t)|^2\,dxdt \leq C_0\int_0^T\int_\omega \left|\sum_{\alpha=1}^{q} w_{\alpha,t}(x,t)\right|^2\,dxdt.$$

Suppose that the claim fails. Then there are initial data in $(H_0^1(\Omega) \times L^2(\Omega))^q$ for which

$$\int_Q \sum_{j=1}^{q} |u_j(x,t)|^2 \, dxdt = 1, \quad \sum_{j=1}^{q} u_{jt}(x,t) = 0 \text{ in } \omega \times (0,T).$$

The contradiction follows from

Unique continuation result.

Lemma

Let ω be an arbitrary nonvoid open subset in Ω. Let T, λ, the constants a_α, and the initial data be given as in Theorem 1. Then

$$\sum_{\alpha=1}^{q} w_{\alpha,t}(x,t) = 0 \text{ in } \omega \times (0,T) \Rightarrow w_\alpha \equiv 0 \text{ in } Q,\ \forall \alpha.$$

The exact controllability of the linear system may be established by relying on the Hilbert Uniqueness Method (HUM) of Lions, or by minimizing the functional

$$
\begin{aligned}
&\mathcal{J} : [L^2(\Omega)]^q \times [H^{-1}(\Omega)]^q \to \mathbb{R}, \text{ given by} \\
&\mathcal{J}(u^0, u^1) = \frac{1}{2}\int_0^T \int_\omega |\sum_{\alpha=1}^q u_\alpha|^2 \, dxdt - \int_\Omega \sum_{\alpha=1}^q z_\alpha^1 u_\alpha(T)\, dx \\
&+ \sum_{\alpha=1}^q \langle u_{\alpha t}(T), z_\alpha^0 \rangle + \sum_{\alpha=1}^q \int_\Omega y_\alpha^1 u_\alpha^0 \, dx - \sum_{\alpha=1}^q \langle u_\alpha^1, y_\alpha^0 \rangle \\
&- \int_Q \sum_{\alpha=1}^q a_\alpha (p_\alpha - f(p_\alpha)) u_\alpha \, dxdt,
\end{aligned}
$$

where $\langle , \rangle$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Theorem 2

Under the hypotheses of Theorem 1, the nonlinear system is exactly controllable; in other words, for $T > T_0 \max\{a_\alpha^{-\frac{1}{2}}; \alpha = 1, 2, ..., q\}$, and for all (y_α^0, y_α^1), and (z_α^0, z_α^1) in $H_0^1(\Omega) \times L^2(\Omega)$, $(\alpha = 1, 2, ..., q)$, there exists a control $v \in L^2(0, T; L^2(\omega))$ such that the corresponding solution of the nonlinear system satisfies the prescribed final conditions.

Proof of Theorem 2: main ideas

- Define the nonlinear mapping $K : [L^2(Q)]^q \to [L^2(Q)]^q$ by $K(p) = y$, where $p = (p_\alpha)_\alpha$ and $y = (y_\alpha)_\alpha$ is the solution of the controlled linearized system. The mapping K is well-defined by Theorem 1.

Proof of Theorem 2: main ideas

- Define the nonlinear mapping $K : [L^2(Q)]^q \to [L^2(Q)]^q$ by $K(p) = y$, where $p = (p_\alpha)_\alpha$ and $y = (y_\alpha)_\alpha$ is the solution of the controlled linearized system. The mapping K is well-defined by Theorem 1.

- K is continuous, and maps any closed ball B_r of $[L^2(Q)]^q$ to B_r

Proof of Theorem 2: main ideas

- Define the nonlinear mapping $K : [L^2(Q)]^q \to [L^2(Q)]^q$ by $K(p) = y$, where $p = (p_\alpha)_\alpha$ and $y = (y_\alpha)_\alpha$ is the solution of the controlled linearized system. The mapping K is well-defined by Theorem 1.

- K is continuous, and maps any closed ball B_r of $[L^2(Q)]^q$ to B_r

- $\overline{K(B_r)}$ is compact in B_r. Finally apply Schauder's fixed-point theorem.

Problem formulation

Let $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}$ be a continuously differentiable function with $f(0,0,0) = 0$, that satisfies for some $p \geq 1$ and some constant $L > 0$:
$\forall s, s', \tau, \tau' \in \mathbb{R},\ \forall \zeta, \zeta' \in \mathbb{R}^N$

$$|f(s,\tau,\zeta) - f(s',\tau',\zeta')| \leq L(|s-s'|(|s|^{p-1} + |s'|^{p-1}) + |\tau - \tau'| + |\zeta - \zeta'|),$$

with $(N-2)p \leq N$.
Given (y_α^0, y_α^1), and (z_α^0, z_α^1) in $H_0^1(\Omega) \times L^2(\Omega)$, $(1 \leq \alpha \leq q)$, find a control function $v \in L^2(0,T; L^2(\omega))$ such that if the q-tuple $(y_\alpha)_{\alpha=1}^q$ solves the system of wave equations:

$$\begin{array}{l} y_{\alpha,tt} - a_\alpha \partial_i (b_{ij}(x) \partial_j y_\alpha) + f(y_\alpha, y_{\alpha t}, \nabla y_\alpha) = v 1_\omega \text{ in } Q \\ y_\alpha = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ y_\alpha(0) = y_\alpha^0; \quad y_{\alpha,t}(0) = y_\alpha^1 \text{ in } \Omega, \end{array}$$

then one has

$$y_\alpha(x,T) = z_\alpha^0(x), \quad y_{\alpha,t}(x,T) = z_\alpha^1 \text{ in } \Omega.$$

Section main result

Theorem 3

Suppose that Ω is of class C^3, and the coefficients b_{ij} lie in $C^2(\bar{\Omega})$. Assume that ω satisfies the Bardos-Lebeau-Rauch geometric control condition. Suppose that the constants a_α, $\alpha = 1,\ 2,\ ...,\ q$, are pairwise distinct. Assume that $\lambda > -\lambda_1$, where λ_1 is the first eigenvalue of $-\Delta_b$ with Dirichlet boundary conditions.

Let $T > T_0 \max\{a_\alpha^{-\frac{1}{2}}; \alpha = 1,\ 2,\ ...,\ q\}$, where T_0 is the controllability time corresponding to the single linear hyperbolic equation, ($f \equiv 0$), with $a_\alpha = 1$. For all (y_α^0, y_α^1), and (z_α^0, z_α^1) in $H_0^1(\Omega) \times L^2(\Omega)$, ($\alpha = 1,\ 2,\ ...,\ q$), with small enough norms, there exists a control $v \in L^2(0, T; L^2(\omega))$ such that the corresponding solution of the nonlinear system satisfies the prescribed final conditions.

Proof of Theorem 3: key elements

- Iterative method.

Proof of Theorem 3: key elements

- Iterative method.
- Let $y_0 = (y_{0\alpha})_\alpha \in [L^\infty(H_0^1(\Omega))]^q \cap [W^{1,\infty}(0, T; L^2(\Omega))]^q$, and for each positive integer n, introduce the system

$$
\begin{aligned}
&y_{n\alpha,tt} - a_\alpha \partial_i(b_{ij}(x)\partial_j y_{n\alpha}) + f(y_{(n-1)\alpha}, y_{(n-1)\alpha,t}, \nabla y_{(n-1)\alpha}) \\
&= v_n 1_\omega \text{ in } Q \\
&y_{n\alpha} = 0 \text{ on } \Sigma \\
&y_{n\alpha}(0) = y_\alpha^0; \quad y_{n\alpha,t}(0) = y_\alpha^1 \text{ in } \Omega.
\end{aligned}
$$

Proof of Theorem 3: key elements

- Iterative method.
- Let $y_0 = (y_{0\alpha})_\alpha \in [L^\infty(H_0^1(\Omega))]^q \cap [W^{1,\infty}(0,T;L^2(\Omega))]^q$, and for each positive integer n, introduce the system

$$\begin{aligned}
&y_{n\alpha,tt} - a_\alpha \partial_i (b_{ij}(x)\partial_j y_{n\alpha}) + f(y_{(n-1)\alpha}, y_{(n-1)\alpha,t}, \nabla y_{(n-1)\alpha}) \\
&= v_n 1_\omega \text{ in } Q \\
&y_{n\alpha} = 0 \text{ on } \Sigma \\
&y_{n\alpha}(0) = y_\alpha^0; \quad y_{n\alpha,t}(0) = y_\alpha^1 \text{ in } \Omega.
\end{aligned}$$

- First, show that for each $n \geq 1$, one can find a control v_n in $L^2(0,T;L^2(\omega))$ such that for every α, the solution $y_n = (y_{n\alpha})_\alpha$ of the linear system satisfies the desired final conditions.

Proof of Theorem 3: key elements

- Iterative method.
- Let $y_0 = (y_{0\alpha})_\alpha \in [L^\infty(H_0^1(\Omega))]^q \cap [W^{1,\infty}(0,T;L^2(\Omega))]^q$, and for each positive integer n, introduce the system

$$
\begin{aligned}
& y_{n\alpha,tt} - a_\alpha \partial_i (b_{ij}(x)\partial_j y_{n\alpha}) + f(y_{(n-1)\alpha}, y_{(n-1)\alpha,t}, \nabla y_{(n-1)\alpha}) \\
& = v_n 1_\omega \text{ in } Q \\
& y_{n\alpha} = 0 \text{ on } \Sigma \\
& y_{n\alpha}(0) = y_\alpha^0; \quad y_{n\alpha,t}(0) = y_\alpha^1 \text{ in } \Omega.
\end{aligned}
$$

- First, show that for each $n \geq 1$, one can find a control v_n in $L^2(0,T;L^2(\omega))$ such that for every α, the solution $y_n = (y_{n\alpha})_\alpha$ of the linear system satisfies the desired final conditions.
- Then prove that the sequence (v_n) converges strongly in $L^2(0,T;L^2(\omega))$ to a function v which is a control for the nonlinear system.

- For every $n \geq 1$, introduce the functional
  $\mathcal{J}_n : [L^2(\Omega)]^q \times [H^{-1}(\Omega)]^q \to \mathbb{R}$, given by

$$\mathcal{J}_n(u^0, u^1) = \frac{1}{2} \int_0^T \int_\omega |\sum_{\alpha=1}^q u_\alpha|^2 \, dxdt - \int_\Omega \sum_{\alpha=1}^q z_\alpha^1 u_\alpha^0 \, dx$$
$$+ \sum_{\alpha=1}^q \langle u_\alpha^1, z_\alpha^0 \rangle + \sum_{\alpha=1}^q \int_\Omega y_\alpha^1 u_\alpha(x,0) \, dx$$
$$- \textstyle\sum_{\alpha=1}^q \langle u_{\alpha,t}(0), y_\alpha^0 \rangle - \displaystyle\int_Q \sum_{\alpha=1}^q f(y_{(n-1)\alpha}, y_{(n-1)\alpha t}, \nabla y_{(n-1)\alpha}) u_\alpha \, dxdt,$$

  where $u = (u_\alpha)_\alpha$ solves the adjoint system

$$\begin{aligned} &u_{\alpha,tt} - a_\alpha \partial_i (b_{ij}(x) \partial_j u_\alpha) = 0 \text{ in } Q \\ &u_\alpha = 0 \text{ on } \Sigma \\ &u_\alpha(T) = u_\alpha^0, \quad u_{\alpha,t}(T) = u_\alpha^1 \text{ in } \Omega. \end{aligned}$$

- For every $n \geq 1$, introduce the functional
  $\mathcal{J}_n : [L^2(\Omega)]^q \times [H^{-1}(\Omega)]^q \to \mathbb{R}$, given by
  $$\mathcal{J}_n(u^0, u^1) = \frac{1}{2} \int_0^T \int_\omega \Big| \sum_{\alpha=1}^q u_\alpha \Big|^2 \, dxdt - \int_\Omega \sum_{\alpha=1}^q z_\alpha^1 u_\alpha^0 \, dx$$
  $$+ \sum_{\alpha=1}^q \langle u_\alpha^1, z_\alpha^0 \rangle + \sum_{\alpha=1}^q \int_\Omega y_\alpha^1 u_\alpha(x,0) \, dx$$
  $$- \textstyle\sum_{\alpha=1}^q \langle u_{\alpha,t}(0), y_\alpha^0 \rangle - \displaystyle\int_Q \sum_{\alpha=1}^q f(y_{(n-1)\alpha}, y_{(n-1)\alpha t}, \nabla y_{(n-1)\alpha}) u_\alpha \, dxdt,$$
  where $u = (u_\alpha)_\alpha$ solves the adjoint system
  $$\begin{aligned} & u_{\alpha,tt} - a_\alpha \partial_i (b_{ij}(x) \partial_j u_\alpha) = 0 \text{ in } Q \\ & u_\alpha = 0 \text{ on } \Sigma \\ & u_\alpha(T) = u_\alpha^0, \quad u_{\alpha,t}(T) = u_\alpha^1 \text{ in } \Omega. \end{aligned}$$
- The functional $\mathcal{J}_n$ is continuous, strictly convex, and coercive. Consequently, $\mathcal{J}_n$ has a unique minimizer $(\hat{u}_n^0, \hat{u}_n^1)$.

- Choose $v_n = \sum_{\alpha=1}^{q} \hat{u}_{n\alpha}$, where $\hat{u}_n = (\hat{u}_{n\alpha})_\alpha$ is the solution of the u-system associated with $(\hat{u}_n^0, \hat{u}_n^1)$.

- Choose $v_n = \sum_{\alpha=1}^{q} \hat{u}_{n\alpha}$, where $\hat{u}_n = (\hat{u}_{n\alpha})_\alpha$ is the solution of the u-system associated with $(\hat{u}_n^0, \hat{u}_n^1)$.
- Use the Euler equation:

$$\begin{aligned}&\int_0^T \int_\omega \sum_{\alpha=1}^{q} \hat{u}_{n\alpha} u_\alpha \, dxdt - \int_\Omega \sum_{\alpha=1}^{q} z_\alpha^1 u_\alpha^0 \, dx + \sum_{\alpha=1}^{q} \langle u_\alpha^1, z_\alpha^0 \rangle \\ &+ \sum_{\alpha=1}^{q} \int_\Omega y_\alpha^1 u_\alpha(x,0) \, dx - \sum_{\alpha=1}^{q} \langle u_{\alpha,t}(0), y_\alpha^0 \rangle \\ &= \int_Q \sum_{\alpha=1}^{q} f(y_{(n-1)\alpha}, y_{(n-1)\alpha,t}, \nabla y_{(n-1)\alpha}) u_\alpha \, dxdt\end{aligned}$$

to show that the sequence of controls (v_n) is bounded:

$$\int_0^T \int_\omega (v_n)^2 \, dxdt \leq C_0(E_0 + E_1) + C_0 L^2 (w_{n-1} + w_{n-1}^p),$$

with $w_n = ||y_n||^2_{L^\infty(0,T;H_0^1(\Omega))} + ||y_{n,t}||^2_{L^\infty(L^2(0,T;\Omega))}$.

- Use energy estimates to derive:

$$w_n \leq C_0(E_0 + E_1) + C_0 L^2 (w_{n-1} + w_{n-1}^p), \quad \forall n \geq 1.$$

- Use energy estimates to derive:

$$w_n \leq C_0(E_0 + E_1) + C_0 L^2 (w_{n-1} + w_{n-1}^p), \quad \forall n \geq 1.$$

- Let $A \geq 1$ be some constant such that

$$w_0 \leq 2A, \quad C_0(E_0 + E_1) \leq A, \quad 2^{p+1} C_0 L^2 A^{p-1} < 1.$$

Use an induction argument to derive that for every $n \geq 1$

$$w_n < 2A, \qquad \int_0^T \int_\omega (v_n)^2 \, dxdt < 2A.$$

- Use energy estimates to derive:

$$w_n \leq C_0(E_0 + E_1) + C_0 L^2 (w_{n-1} + w_{n-1}^p), \quad \forall n \geq 1.$$

- Let $A \geq 1$ be some constant such that

$$w_0 \leq 2A, \quad C_0(E_0 + E_1) \leq A, \quad 2^{p+1} C_0 L^2 A^{p-1} < 1.$$

  Use an induction argument to derive that for every $n \geq 1$

$$w_n < 2A, \qquad \int_0^T \int_\omega (v_n)^2 \, dxdt < 2A.$$

- Use those estimates and the Euler equation once more to show that (v_n) is a Cauchy sequence in $L^2(0, T; L^2(\omega)$ while (y_n) is a Cauchy sequence in $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$.

- Use energy estimates to derive:

$$w_n \leq C_0(E_0 + E_1) + C_0 L^2(w_{n-1} + w_{n-1}^p), \quad \forall n \geq 1.$$

- Let $A \geq 1$ be some constant such that

$$w_0 \leq 2A, \quad C_0(E_0 + E_1) \leq A, \quad 2^{p+1} C_0 L^2 A^{p-1} < 1.$$

  Use an induction argument to derive that for every $n \geq 1$

$$w_n < 2A, \qquad \int_0^T \int_\omega (v_n)^2 \, dxdt < 2A.$$

- Use those estimates and the Euler equation once more to show that (v_n) is a Cauchy sequence in $L^2(0, T; L^2(\omega)$ while (y_n) is a Cauchy sequence in $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$.
- Derive that (v_n) converges to a certain function v in $L^2(0, T; L^2(\omega))$, which is a control for the nonlinear system.

- Systems of uncoupled wave equations arise from the study of networks of strings (one-dimensional setting), or from the linearized models of chemical reactions with diffusion in multicomponent systems (J. Chem. Phys. 67 (1977), 3382.) They were introduced in the control framework by Russell in his investigation of the boundary controllability of the Maxwell equations.

- Systems of uncoupled wave equations arise from the study of networks of strings (one-dimensional setting), or from the linearized models of chemical reactions with diffusion in multicomponent systems (J. Chem. Phys. 67 (1977), 3382.) They were introduced in the control framework by Russell in his investigation of the boundary controllability of the Maxwell equations.
- Haraux (1988) showed for arbitrary ω:
  - If $\sum_{\alpha=1}^{q} u_\alpha(x,t) = 0$ in $\omega \times (0,T)$ then $u_\alpha^0 = 0, \quad u_\alpha^1 = 0$ in $\Omega, \quad \forall \alpha$.
  - If $N = 1$ and T is large enough, or $\omega = \Omega$, then for all j and all $(u_\alpha^0, u_\alpha^1) \in L^2(\Omega) \times H^{-1}(\Omega)$:

$$\sum_{\alpha=1}^{q} \{||u_\alpha^0||_{L^2(\Omega)}^2 + ||u_\alpha^1||_{H^{-1}(\Omega)}^2\} \leq C \int_0^T \int_\omega |\sum_{\alpha=1}^{q} u_\alpha(x,t)|^2 \, dxdt.$$

- The analogues of Theorems 1, 3 hold for the plate equations with various boundary conditions.

- The analogues of Theorems 1, 3 hold for the plate equations with various boundary conditions.
- Extending the observability estimate to the case of a potential depending on both the space and time variables is open.

- The analogues of Theorems 1, 3 hold for the plate equations with various boundary conditions.
- Extending the observability estimate to the case of a potential depending on both the space and time variables is open.
- Extending the results to the boundary controllability setting is widely open in higher space dimensions; apparently exact boundary controllability in the standard energy space is not to be expected according to the one-dimensional observability estimate of e.g. Komornik-Loreti, (Chap. 9, book (2005)).

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!