On some stabilization problems for the Timoshenko beam

Louis Tebou

Florida International University

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Overview

- The simultaneously damped system.

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- The indirectly damped system.

Problem formulation

Let $L > 0$, and set $\Omega = (0, L)$, and $\omega = (l_1, l_2)$ with $0 \leq l_1 < l_2 \leq L$. Consider the damped Timoshenko system:

$$\begin{cases} \rho_1 y_{tt} - k(y_x + z)_x + a(x)(y_t + z_t) = 0 \text{ in } (0, L) \times (0, \infty) \\ \rho_2 z_{tt} - \sigma z_{xx} + k(y_x + z) + a(x)(y_t + z_t) = 0 \text{ in } (0, L) \times (0, \infty), \end{cases}$$

with the boundary conditions:

(**DD**) $\quad y(0, t) = 0, \quad y(L, t) = 0, \quad z(0, t) = 0, \quad z(L, t) = 0,$ or else

(**DN**) $\quad y(0, t) = 0, \quad y(L, t) = 0, \quad z_x(0, t) = 0, \quad z_x(L, t) = 0, \quad t > 0$

and the initial conditions:

$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x), \quad x \in \Omega.$

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The damping coefficient a is a nonnegative bounded measurable function, which is positive in ω only.

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The fact that this damping is degenerate makes the stabilization problem more challenging to analyze.

The energy and main questions

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{\rho_1|y_t(x,t)|^2 + k|y_x(x,t) + z(x,t)|^2\}\,dx \\ +\frac{1}{2}\int_\Omega \{\rho_2|z_t(x,t)|^2 + \sigma|z_x(x,t)|^2\}\,dx, \quad \forall t \geq 0.$$

The energy E is a nonincreasing function of the time variable t as we have for every $t \geq 0$, (hereafter, $'$ denotes differentiation with respect to time)

$$E'(t) = -\int_\Omega a(x)|y_t(x,t) + z_t(x,t)|^2\,dx.$$

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Our main purpose in this section of the talk is to answer the following questions:

- Does the energy $E(t)$ decay to zero as the time variable t goes to infinity?
- If so, how fast? And if not, why?

An abstract framework

Set $Z = \begin{pmatrix} y \\ y' \\ z \\ z' \end{pmatrix}$. Our initial system may then be recast as:

$$Z' - \mathcal{A}Z = 0 \text{ in } (0, \infty), \qquad Z(0) = \begin{pmatrix} y^0 \\ y^1 \\ z^0 \\ z^1 \end{pmatrix}$$

where the unbounded operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{k}{\rho_1}\partial_x^2 & -\frac{a}{\rho_1}I & \frac{k}{\rho_1}\partial_x & -\frac{a}{\rho_1}I \\ 0 & 0 & 0 & I \\ -\frac{k}{\rho_2}\partial_x & -\frac{a}{\rho_2}I & \frac{\sigma}{\rho_2}\partial_x^2 - \frac{k}{\rho_2}I & -\frac{a}{\rho_2}I \end{pmatrix}$$

In the (**DD**) case:
$D(\mathcal{A}) = \Big\{(u,v,w,z) \in \big(H_0^1(\Omega) \times H_0^1(\Omega)\big)^2 ; k(u_x + w)_x - a(v+z) \in L^2(\Omega),\ \text{and } \sigma w_{xx} - k(u_x + w) - a(v+z) \in L^2(\Omega)\Big\}$ and, in the (**DN**) case:
$D(\mathcal{A}) = \Big\{(u,v,w,z) \in (H_0^1(\Omega))^2 \times V^2 ; k(u_x + w)_x - a(v+z) \in L^2(\Omega),\ \text{and } \sigma w_{xx} - k(u_x + w) - a(v+z) \in L^2(\Omega)\Big\}$
$V = \{u \in H^1(\Omega); \int_\Omega u(x)\,dx = 0\}$.
One checks that in the (**DD**) case, one has
$D(\mathcal{A}) = \big((H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)\big)^2$
In the (**DN**) case, $D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap V) \times V$.
Thus, in either case, the operator $\mathcal{A}$ has a compact resolvent.
Consequently the spectrum of $\mathcal{A}$ is discrete.

Introduce the Hilbert spaces over the field $\mathbb{C}$ of complex numbers $\mathcal{H}_1 = \left(H_0^1(\Omega) \times L^2(\Omega)\right)^2$ and $\mathcal{H}_2 = H_0^1(\Omega) \times L^2(\Omega) \times V \times L^2(\Omega)$, equipped with the norm

$$||Z||^2_{\mathcal{H}_i} = \int_\Omega \{\rho_1 |v|^2 + k|u_x + w|^2 + \rho_2 |z|^2 + \sigma |w_x|^2\}\, dx,$$

$$\forall Z = (u, v, w, z) \in \mathcal{H}_i, \quad i = 1,\ 2.$$

Main results and proof ideas

Theorem 1

Suppose that ω is an arbitrary nonempty open interval in Ω. Let the damping coefficient a be positive in ω. In either of the (**DD**) or (**DN**) case, the associated operator $\mathcal{A}$ generates a C_0 semigroup of contractions $(S_i(t))_{t\geq 0}$ on the corresponding Hilbert space $\mathcal{H}_i$, (i=1, 2), which is strongly stable:

$$\lim_{t\to\infty} ||S_i(t)Z^0||_{\mathcal{H}} = 0, \quad \forall Z^0 \in \mathcal{H}_i,$$

if and only if $\partial\omega \cap \partial\Omega \neq \emptyset$.

Proof of Theorem 1: key elements

- Semigroup generation: pretty straightforward thanks to Lumer-Phillips Theorem.
- Strong stability: Thanks to a Benchimol result, it suffices to show that $\mathcal{A}$ has no imaginary eigenvalue.

One easily checks that zero is not an eigenvalue of $\mathcal{A}$. Now, let b be a nonzero real number, and let $Z = (u, v, w, z) \in D(\mathcal{A})$ such that $\mathcal{A}Z = ibZ$. We shall prove that $Z = (0, 0, 0, 0)$ if and only if $\partial\omega \cap \partial\Omega \neq \emptyset$. Note that $\mathcal{A}Z = ibZ$ may be recast as:

$$
\begin{aligned}
&-b^2 u - \hat{k}(u_x + w)_x + ib\hat{a}(u + w) = 0 \text{ in } (0, L) \\
&-b^2 w - \hat{\sigma} w_{xx} + \check{k}(u_x + w) + ib\check{a}(u + w) = 0 \text{ in } (0, L).
\end{aligned}
$$

One easily checks that $u = -w$ in ω.

The heart of the matter now is to check under which condition, one has $u \equiv 0$ in ω; indeed, if we can prove that $u \equiv 0$ in ω, then $w \equiv 0$ in ω, and basic uniqueness results may then be invoked to conclude that $Z = (0, 0, 0, 0)$. Therefore, it remains to find under which condition $u \equiv 0$ in ω. Since $u = -w$ in ω, our system reduces to:

$$
\begin{aligned}
-b^2 u - \hat{k}(u_x - u)_x = 0 \text{ in } \omega \\
b^2 u + \hat{\sigma} u_{xx} + \check{k}(u_x - u) = 0 \text{ in } \omega.
\end{aligned}
$$

Adding both equations, one gets rid of b, thereby obtaining

$$
(\hat{\sigma} - \hat{k})u_{xx} + (\hat{k} + \check{k})u_x - \check{k}u = 0 \text{ in } \omega.
$$

The characteristic equation for this equation is is given by:

$$(\hat{\sigma} - \hat{k})r^2 + (\hat{k} + \check{k})r - \check{k} = 0.$$

Using the latter equation, one shows that $u \equiv 0$ in ω when $\partial\omega \cap \partial\Omega \neq \emptyset$, and that the operator $\mathcal{A}$ does have imaginary eigenvalues if $\partial\omega \cap \partial\Omega = \emptyset$. Hence the associated semigroup is strongly stable if and only if $\partial\omega \cap \partial\Omega \neq \emptyset$.

Theorem 2

Suppose that ω is an arbitrary nonempty open interval in Ω with $\partial\omega \cap \partial\Omega \neq \emptyset$. Let the damping coefficient a satisfy

$$a(x) \geq a_0 > 0, \text{ a.e. in } \omega.$$

For each $i = 1,\ 2$, the semigroup $(S_i(t))_{t\geq 0}$ is exponentially stable, viz., there exist positive constants M and λ with

$$||S_i(t)Z^0||_{\mathcal{H}_i} \leq M \exp(-\lambda t)||Z^0||_{\mathcal{H}_i}, \quad \forall Z^0 \in \mathcal{H}_i.$$

Proof of Theorem 2: main ideas

Set $\omega = (l_1, L)$. We focus on the (**DD**) case, and set $\mathcal{H} = \mathcal{H}_1$. Thanks to results due to Pruss, and Huang, it suffices to show:

1. $i\mathbb{R} \subset \rho(\mathcal{A})$
2. $\sup\{||(ib - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})};\ b \in \mathbb{R}\} < \infty$

The first point follows from the proof of Theorem 1.
Proving the second point amounts to showing:

$$\exists C_0 > 0 : \forall U \in \mathcal{H},\ \text{one has } ||(ib - \mathcal{A})^{-1}U||_{\mathcal{H}} \leq C_0||U||_{\mathcal{H}}, \quad \forall b \in \mathbb{R}.$$

Now, let $b \in \mathbb{R}$, $U = (f, g, h, l) \in \mathcal{H}$, and let $Z = (u, v, w, z) \in D(\mathcal{A})$ such that

$$(ib - \mathcal{A})Z = U.$$

We shall prove:

$$||Z||_{\mathcal{H}} \leq C_0||U||_{\mathcal{H}}.$$

Our initial estimate:

$$\int_\Omega a(x)|v(x)+z(x)|^2\,dx = \Re(U,Z) \le ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}.$$

Recast the U-Z equation as:

$$\begin{aligned}
&ibu - v = f\\
&ibv - \hat{k}(u_x + w)_x + \hat{a}(v+z) = g\\
&ibw - z = h\\
&ibz - \hat{\sigma} w_{xx} + \check{k}(u_x + w) + \check{a}(v+z) = l.
\end{aligned}$$

Using appropriate multipliers, we arrive at (for large enough $|b|$)

$$\check{k}b^2|u|_2^2 + \check{k}\hat{k}|u_x + w|_2^2 + \hat{\sigma}b^2|w|_2^2 + \hat{\sigma}^2|w_x|_2^2$$
$$\leq C_0 b^2 \int_\Omega \eta^2(|u|^2 + |w|^2)\,dx + C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^2)$$

$$\leq C_0 b^2 \int_\Omega \eta^2|u+w|^2\,dx - 2C_0 b^2 \Re \int_\Omega \eta^2 u\bar{w}\,dx$$

$$+ C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^2).$$

Resorting once more to adequate multipliers, we find

$$-b^2(\hat{\sigma} - \hat{k})\Re \int_\Omega \eta^2 u\bar{w}\,dx = \hat{\sigma}\Re \int_\Omega ((\hat{g} + ibf)\eta^2 - 2\hat{k}\eta_x\eta(u_x + w))\bar{w}\,dx$$

$$-\hat{k}\Re \int_\Omega ((\hat{l} + ibh - \check{k}(u_x + w))\eta^2\bar{u} - 2\hat{\sigma}\eta_x\eta w_x\bar{u} + \hat{\sigma}\bar{w}_x w)\,dx.$$

If $\hat{k} \neq \hat{\sigma}$, one then gets the estimate:

$$\check{k} b^2 |u|_2^2 + \check{k}\hat{k} |u_x + w|_2^2 + \hat{\sigma} b^2 |w|_2^2 + \hat{\sigma}^2 |w_x|_2^2$$
$$\leq C_0(||U||_{\mathcal{H}} ||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^2).$$

If $\hat{k} = \hat{\sigma}$, we need a different approach. Using the multiplier $\eta^2 \bar{w}$, we derive

$$-b^2 \Re \int_\Omega \eta^2 u \bar{w}\, dx = -\hat{k} \Re \int_\Omega \eta^2 u_x \bar{w}_x\, dx$$

$$-\hat{k} \Re \int_\Omega (\eta^2 w_x + 2\eta_x \eta (u_x + w)) \bar{w}\, dx + \Re \int_\Omega (\hat{g} + ibf) \eta^2 \bar{w}\, dx.$$

Introduce $\varphi = u + w$, which solves:

$$-b^2 \varphi - \hat{k} \varphi_{xx} = \hat{g} + \hat{l} + ib(f + h) + \hat{k} w_x - \check{k}(u_x + w) \text{ in } \Omega.$$

Using that equation, one checks that

$$\int_\Omega \eta^2|\varphi_x|^2\,dx \le C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||^2_{\mathcal{H}} + ||U||^{\frac{1}{2}}_{\mathcal{H}}||Z||^{\frac{3}{2}}_{\mathcal{H}}).$$

With the help of the multiplier $\eta^2\bar{w}_x$, we derive from the same equation:

$$\begin{aligned}
&\hat{k}\int_\Omega \eta^2|w_x|^2\,dx - \check{k}\Re\int_\Omega \eta^2 u_x\bar{w}_x\,dx \\
&= -\Re\int_\Omega(\eta^2 h_x + 2\eta_x\eta(ib\bar{w} + \bar{h}))ib\varphi\,dx - \hat{k}\Re\varphi_x(L)\overline{w_x(L)} \\
&\quad +\Re\int_\Omega \eta^2\varphi_x(\check{k}(\bar{u}_x + \bar{w}) - \bar{\hat{l}})\,dx \\
&\quad -\Re\int_\Omega \eta^2(\hat{g} + \hat{l} - \check{k}w)\bar{w}_x - ib(\eta^2(f+h))_x\bar{w}\,dx \\
&\le C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||^{\frac{1}{2}}_{\mathcal{H}}||Z||^{\frac{3}{2}}_{\mathcal{H}} + ||U||^2_{\mathcal{H}} + ||U||^{\frac{1}{4}}_{\mathcal{H}}||Z||^{\frac{7}{4}}_{\mathcal{H}} + ||U||^{\frac{1}{8}}_{\mathcal{H}}||Z||^{\frac{15}{8}}_{\mathcal{H}}) \\
&\quad +C_0|w|_2^2.
\end{aligned}$$

On the other hand, the estimate of φ_x leads to:

$$\int_\Omega \eta^2|w_x|^2\,dx + \Re\int_\Omega \eta^2 u_x\bar{w}_x\,dx = \Re\int_\Omega \eta^2\varphi_x\bar{w}_x\,dx$$
$$\le C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^{\frac{1}{4}}||Z||_{\mathcal{H}}^{\frac{7}{4}}).$$

It now follows from the last two inequalities:

$$\left|\Re\int_\Omega \eta^2 u_x\bar{w}_x\,dx\right|$$
$$\le C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^{2} + ||U||_{\mathcal{H}}^{\frac{1}{4}}||Z||_{\mathcal{H}}^{\frac{7}{4}} + ||U||_{\mathcal{H}}^{\frac{1}{8}}||Z||_{\mathcal{H}}^{\frac{15}{8}})$$
$$+C_0|w|_2^2.$$

Hence

$$\check{k}b^2|u|_2^2 + \check{k}\hat{k}|u_x+w|_2^2 + \hat{\sigma}b^2|w|_2^2 + \hat{\sigma}^2|w_x|_2^2$$
$$\le C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^{2} + ||U||_{\mathcal{H}}^{\frac{1}{4}}||Z||_{\mathcal{H}}^{\frac{7}{4}} + ||U||_{\mathcal{H}}^{\frac{1}{8}}||Z||_{\mathcal{H}}^{\frac{15}{8}}).$$

Problem formulation

Consider the new Timoshenko system:

$$\begin{cases} \rho_1 y_{tt} - k(y_x + z)_x + a(x) y_t = 0 \text{ in } (0, L) \times (0, \infty) \\ \rho_2 z_{tt} - \sigma z_{xx} + k(y_x + z) = 0 \text{ in } (0, L) \times (0, \infty). \end{cases}$$

Remark. Many stabilization problems for the indirectly damped Timoshenko system involve the case where the damping appears in the shear angle equation z, and exponential stability holds if and only if $\hat{k} = \hat{\sigma}$. This latter condition is never satisfied in the physically relevant problem. So, for this reason, some authors have proposed using two independent damping mechanisms, one in the transverse displacement equation and the other one in the shear angle equation. Using two independent feedback controls simplify the mathematical analysis. It is then a natural question to wonder what would happen if one were to use a single control, but now acting through the bending equation instead.

Findings for the new system

- When $\omega = \Omega$, exponential stability holds if and only if $\hat{k} = \hat{\sigma}$, whether we have (**DD**) or (**DN**).
- One notes that in the (**DN**) case, even strong stability fails if $\omega \neq \Omega$.
- In the (**DD**) case, when $\omega \neq \Omega$, strong stability holds if ω meets an endpoint of Ω, and exponential decay holds if further $\hat{k} = \hat{\sigma}$.

One notes here that the boundary conditions play a role in the stability analysis, while they did not matter in earlier works.

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!