# Assignment 2 – key

1. $u_1=(2,1,3)^T$, $u_2=(1,m,1)^T$, $u_3=(-1,1,-m)^T$ are linearly independent if $\begin{vmatrix} 2 & 1 & -1 \\ 1 & m & 1 \\ 3 & 1 & -m \end{vmatrix} \neq 0$. Now $\begin{vmatrix} 2 & 1 & -1 \\ 1 & m & 1 \\ 3 & 1 & -m \end{vmatrix} \xrightarrow[2C_3+C_1]{C_3+C_2} \begin{vmatrix} 0 & 0 & -1 \\ 3 & m+1 & 1 \\ 3-2m & 1-m & -m \end{vmatrix}$

$= -[3(1-m)-(m+1)(3-2m)]$
$= -[3-3m-(3m+3-2m^2-2m)]$
$= -[\cancel{3}-3m-m-\cancel{3}+2m^2$
$= -2m^2+4m$
$= -2m(m-4)$

So $u_1, u_2, u_3$ are linearly independent for $m \neq 0$ and $m \neq 4$.

2. For which values of $a$ and $b$ can we find $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ scalars: $V=\alpha_1V_1+\alpha_2V_2+\alpha_3V_3+\alpha_4V_4$? If $X=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^T$ and $A$ denotes the matrix with columns $V_1, V_2, V_3$ and $V_4$, this problem amounts to finding the values of $a$ and $b$ for which the linear system $AX=V$ is consistent. We use the Gauss reduction process.

$$\left(\begin{array}{cccc|c} 1 & 2 & 6 & 1 & 2 \\ 2 & 5 & 17 & 3 & 4 \\ -4 & -3 & -7 & -3 & 6 \\ 3 & 4 & 10 & 2 & a \\ 1 & 8 & 22 & 0 & b \end{array}\right) \xrightarrow[\substack{4r_1+r_3 \\ -3r_1+r_4 \\ -r_1+r_5}]{-2r_1+r_2} \left(\begin{array}{cccc|c} 1 & 2 & 6 & 1 & 2 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 5 & 17 & 1 & 14 \\ 0 & -2 & -8 & -1 & a-6 \\ 0 & 6 & 16 & -1 & b-2 \end{array}\right) \xrightarrow[\substack{2r_2+r_4 \\ -6r_2+r_5}]{-5r_2+r_3}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 6 & 1 & 2 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & -8 & -4 & 14 \\ 0 & 0 & 2 & 1 & a-6 \\ 0 & 0 & -14 & -7 & b-2 \end{array}\right) \xrightarrow{r_3 \longleftrightarrow r_4} \left(\begin{array}{cccc|c} 1 & 2 & 6 & 1 & 2 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 2 & 1 & a-6 \\ 0 & 0 & -8 & -4 & 14 \\ 0 & 0 & -14 & -7 & b-2 \end{array}\right) \xrightarrow[7r_3+r_5]{4r_3+r_4}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 6 & 1 & 2 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 2 & 1 & a-6 \\ 0 & 0 & 0 & 0 & 14+4(a-6) \\ 0 & 0 & 0 & 0 & 7(a-6)+b-2 \end{array}\right)$$

This system is consistent if and only if $-10+4a=0$, and $7a+b-44=0$.

$a=5/2$ and $b=44-\frac{35}{2}=\frac{53}{2}$.

Thus $V$ belongs to $\text{Span}(V_1,V_2,V_3,V_4)$ iff $a=\frac{5}{2}$ and $b=\frac{53}{2}$.

3. a) Thanks to Theorem 3.4.3, we shall show $\det((u_1, u_2, u_3)) \neq 0$ and $\det((v_1, v_2, v_3)) \neq 0$.

$$\det((u_1,u_2,u_3)) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -2 & 2 \\ 3 & 1 & 1 \end{vmatrix} = (-2-2)-(2-6)+2(2+6) = -4+4+16 \neq 0$$

So $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$.

$$\det((v_1,v_2,v_3)) = \begin{vmatrix} 1 & 5 & 1 \\ 2 & -3 & 1 \\ 1 & 4 & 1 \end{vmatrix} = (-3-4)-5(2-1)+(8+3) = -7-5+11 \neq 0$$

So $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

b) If we set $B = [u_1, u_2, u_3]$, $D = [v_1, v_2, v_3]$, $U = (u_1, u_2, u_3)$, $V = (v_1, v_2, v_3)$. We shall find $T_{B \to D} = V^{-1}U$; so we start with the augmented matrix $(V \mid U)$ and get its RREF.

$$\left(\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 1 & 2 \\ 2 & -3 & 1 & 2 & -2 & 2 \\ 1 & 4 & 1 & 3 & 1 & 1 \end{array}\right) \xrightarrow[-r_1+r_3]{-2r_1+r_2} \left(\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 1 & 2 \\ 0 & -13 & -1 & 0 & -4 & -2 \\ 0 & -1 & 0 & 2 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{r_2 \longleftrightarrow r_3} \left(\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -13 & -1 & 0 & -4 & -2 \end{array}\right) \xrightarrow[-13r_2+r_3]{5r_2+r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 11 & 1 & -3 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & -26 & -4 & 11 \end{array}\right)$$

$$\xrightarrow[\substack{-r_2 \\ -r_3}]{r_3+r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -15 & -3 & 8 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 26 & 4 & -11 \end{array}\right)$$

Hence $T_{B \to D} = \begin{pmatrix} -15 & -3 & 8 \\ -2 & 0 & 1 \\ 26 & 4 & -11 \end{pmatrix}$

c) If $u = 5u_1 + 2u_2 - 4u_3$, then $u = T_{B\to D}\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -113 \\ -14 \\ 182 \end{pmatrix}_D$. So $u = -113v_1 - 14v_2 + 182v_3$.

4. a) $R(A+B) = \{z = Ax + Bx,$ for some $x \in \mathbb{R}^n\}$

$z \in R(A+B) \Rightarrow z = Ax + Bx$ for some $x \in \mathbb{R}^n$

Now $Ax \in R(A)$ and $Bx \in R(B)$. So $Ax + Bx \in R(A) + R(B)$. Hence

$R(A+B) \subseteq R(A) + R(B)$.

b) If $r_{A+B}$ denotes the rank of $A+B$, then $r_{A+B} = n$, since $A+B$ is nonsingular. Since $R(A+B) \subseteq R(A) + R(B)$, we have

$r_{A+B} \leq \dim(R(A) + R(B)) \leq \dim R(A) + \dim R(B) = r_A + r_B$; So

$n \leq r_A + r_B$. Let's show now $r_A + r_B \leq n$.

Since $AB = O_{Mn}$, it follows from problem solved in class that $R(B) \subseteq N(A)$; so $r_B \leq n_A$. Now $n = r_A + n_A$ by the rank-nullity theorem; hence $n \geq r_A + r_B$, as $n_A \geq r_B$.

Consequently $r_A + r_B = n$.