Assignment 3 – key

1. a) $\det(A-\lambda I_3) = \begin{vmatrix} 4-\lambda & 1 & -2 \\ 0 & 5-\lambda & -2 \\ 3 & 1 & -1-\lambda \end{vmatrix} = (4-\lambda)[(5-\lambda)(-1-\lambda)+2] + 3[-2+2(5-\lambda)]$

$= (4-\lambda)((5-\lambda)(-1-\lambda)) + 8 - 2\lambda + 24 - 6\lambda$

$= (4-\lambda)(5-\lambda)(-1-\lambda) + 32 - 8\lambda$

$= (4-\lambda)(5-\lambda)(-1-\lambda) + 8(4-\lambda)$

$= (4-\lambda)[-5-4\lambda+\lambda^2+8]$

$= (4-\lambda)(\lambda^2-4\lambda+3)$

$= (4-\lambda)(\lambda-3)(\lambda-1)$

$= 0 \longrightarrow \lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 4$

b) • For $\lambda = \lambda_1 = 1$, solve $(A - I_3)x = 0$,

$$\left(\begin{array}{ccc|c} 3 & 1 & -2 & 0 \\ 0 & 4 & -2 & 0 \\ 3 & 1 & -2 & 0 \end{array}\right) \xrightarrow{-r_1+r_3} \left(\begin{array}{ccc|c} 3 & 1 & -2 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{l} 3x_1+x_2-2x_3=0 \\ 2x_2-x_3=0 \end{array}$$

$x_3 = 2x_2$, $x_1 = -\frac{x_2}{3} + \frac{2x_3}{3} = -\frac{x_2}{3} + \frac{4x_2}{3} = x_2$

$E_1 = \text{span}((1,1,2)^T)$

• For $\lambda = \lambda_2 = 3$, $(A - 3I_3)X = 0$,

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 3 & 1 & -4 & 0 \end{array}\right) \xrightarrow{-3r_1+r_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \xrightarrow{r_2+r_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{l} x_1+x_2-2x_3=0 \longrightarrow x_1 = -x_2+2x_3 = x_3 \\ x_2-x_3=0 \longrightarrow x_2 = x_3 \end{array}$$

$E_3 = \text{span}((1,1,1)^T)$

• For $\lambda = \lambda_3 = 4$, $(A - 4I_3)X = 0$,

$$\begin{pmatrix} 0 & 1 & -2 \\ 0 & 1 & -2 \\ 3 & 1 & -5 \end{pmatrix} \xrightarrow{-r_1+r_2} \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 3 & 1 & -5 \end{pmatrix} \longrightarrow \begin{array}{l} x_2-2x_3=0 \\ 3x_1+x_2-5x_3=0 \end{array}$$

$x_2 = 2x_3$

$3x_1 + 2x_3 - 5x_3 = 0 \longrightarrow x_1 = x_3$

$E_4 = \text{span}((1,2,1)^T)$

c) $A = PDP^{-1}$, $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$P^{-1} = \text{adj}P = \begin{pmatrix} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \end{pmatrix}$

$\det(P) = 1(1-2) - 1(1-4) + 1(1-2)$
$= -1 + 3 - 1$
$= 1.$

$P^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 3 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$, $\underline{\underline{U = P}}$

d) Set $B = PD^{1/5}P^{-1}$, then $B^5 = PDP^{-1} = A$.

$$D^{1/5} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt[5]{3} & 0 \\ 0 & 0 & \sqrt[5]{4} \end{pmatrix}.$$

2) a) $N^2 = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 3 & -6 \\ -3 & 6 \end{pmatrix} = 3N$, $N^3 = N^2N = 3N^2 = 3(3N) = 3^2N$

$N^4 = N^3(N) = 3^2N^2 = 3^2(3N) = 3^3N$. There is a pattern

b) $N^n = 3^{n-1}N$, $n = 1, 2, \cdots$

<u>Proof</u>. By induction.

<u>Base step</u>; $n=1$: $N^1 = 3^0N$; formula is valid for $N=1$, as $3^0=1$.

<u>Inductive Hypothesis</u>. Let $n \geq 1$. Suppose $N^n = 3^{n-1}N$.

Show $N^{n+1} = 3^nN$.

$$\begin{aligned} N^{n+1} &= N^nN \\ &= (3^{n-1}N)N, \text{ by Inductive Hypothesis} \\ &= 3^{n-1}N^2 \\ &= 3^{n-1}(3N), \text{ by a)} \\ &= 3^nN. \end{aligned}$$

Hence formula holds for all $n \geq 1$.

c) $e^N = \sum_{k=0}^{\infty} \frac{N^k}{k!} = I_2 + \sum_{k=1}^{\infty} \frac{N^k}{k!} = I_2 + \sum_{k=1}^{\infty} \frac{3^{k-1}}{k!}N = I_2 + \underbrace{\left(\sum_{k=1}^{\infty} \frac{3^k}{k!}\right)}_{\left[\left(\sum_{k=0}^{\infty} \frac{3^k}{k!}\right) - 1\right]} \frac{N}{3}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}(e^3 - 1)$$

$$= \begin{pmatrix} \frac{e^3+2}{3} & -\frac{2(e^3-1)}{3} \\ -\frac{(e^3-1)}{3} & \frac{(e^3+1)}{3} \end{pmatrix}$$

4) $A \in M_{m,n}$, $r_A = n$. Set $P = A(A^TA)^{-1}A^T$.

a) $y \in R(P) \Rightarrow y = Px$ for some $x \in \mathbb{R}^m$

$$\begin{aligned} &= A(A^TA)^{-1}A^Tx \\ &= A[(A^TA)^{-1}A^Tx] \in R(A); \text{ so } R(P) \subseteq R(A) \end{aligned}$$

$z \in R(A) \Rightarrow z = Au$, for some $u \in \mathbb{R}^n$

$$\begin{aligned} \Rightarrow z &= A((A^TA)^{-1}(A^TA)u) \\ &= (A(A^TA)^{-1}A^T)Au \\ &= P(Au) \in R(P); \text{ so } R(A) \subseteq R(P). \text{ Hence } R(A) = R(P). \end{aligned}$$

b) Check $P^T = P$. $R(P)^\perp = R(A)^\perp$, by a)

$N(P^T) = N(A^T)$, by Theorem 5.2.1

$N(P) = N(A^T)$.

3. a) First, find a basis for $R(M)$, then use the Gram-Schmidt process.

$$\begin{pmatrix}1&1&1\\2&1&1\\-2&-1&-1\\4&2&3\end{pmatrix} \xrightarrow[\substack{2r_1+r_3\\-4r_1+r_4}]{-2r_1+r_2} \begin{pmatrix}1&1&1\\0&-1&-1\\0&1&1\\0&-2&-1\end{pmatrix} \xrightarrow[-2r_2+r_4]{r_2+r_3} \begin{pmatrix}1&1&1\\0&-1&-1\\0&0&0\\0&0&1\end{pmatrix} \xrightarrow[r_3\leftrightarrow r_4]{-r_2}$$

$$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\\0&0&0\end{pmatrix}. \text{ So } R(M) = \mathrm{Span}\Big(\underbrace{\begin{pmatrix}1\\2\\-2\\4\end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix}1\\1\\-1\\2\end{pmatrix}}_{u_2}, \underbrace{\begin{pmatrix}1\\1\\-1\\3\end{pmatrix}}_{u_3}\Big)$$

$\{u_1, u_2, u_3\}$ is a basis for $R(M)$. Now apply G-S.

Set $q_1 = \dfrac{u_1}{\|u_1\|_2} = \dfrac{u_1}{5}$

$$q_2 = \frac{u_2 - \langle u_2, q_1\rangle q_1}{\|u_2 - \langle u_2, q_1\rangle q_1\|_2}, \quad \langle u_2, q_1\rangle = \frac{1+2+2+8}{5} = \frac{13}{5}$$

$$= \frac{u_2 - \frac{13 u_1}{25}}{\|u_2 - \frac{u_1(13)}{25}\|_2}$$

$$= \frac{(12/25, -1/25, 1/25, -2/25)^T}{\frac{1}{25}\sqrt{12^2+1+1+4}} = \frac{(12,-1,1,-2)^T}{5\sqrt{6}}$$

$$q_3 = \frac{u_3 - \langle u_3, q_1\rangle q_1 - \langle u_3, q_2\rangle q_2}{\|u_3 - \langle u_3, q_1\rangle q_1 - \langle u_3, q_2\rangle q_2\|_2}, \quad \langle u_3, q_1\rangle = \frac{1+2+2+12}{5} = \frac{17}{5}, \quad \langle u_3, q_2\rangle = \frac{12-1-1-6}{5\sqrt{6}} = \frac{4}{5\sqrt{6}}$$

$$= \frac{(0, -1/3, 1/3, 1/3)}{\sqrt{1/3}}$$

$\{q_1, q_2, q_3\}$ is an orthonormal basis for $R(M)$

b) Set $Q = (q_1, q_2, q_3) = \begin{pmatrix}1/5 & 12/5\sqrt{6} & 0\\ 2/5 & -1/5\sqrt{6} & -1/\sqrt{3}\\ -2/5 & 1/5\sqrt{6} & 1/\sqrt{3}\\ 4/5 & -2/5\sqrt{6} & 1/\sqrt{3}\end{pmatrix}$,

$r_{11} = \|u_1\| = 5$
$r_{12} = \langle u_2, q_1\rangle = 13/5$
$r_{13} = \langle u_3, q_1\rangle = 17/5$
$r_{22} = \|u_2 - \langle u_2, q_1\rangle q_1\|_2 = \sqrt{6}/5$
$r_{23} = \langle u_3, q_2\rangle = 4/5\sqrt{6}$
$r_{33} = \|u_3 - \langle u_3, q_1\rangle q_1 - \langle u_3, q_2\rangle q_2\|_2 = \sqrt{1/3} = 1/\sqrt{3}$

$$R = \begin{pmatrix}5 & 13/5 & 17/5\\ 0 & \sqrt{6}/5 & 4/5\sqrt{6}\\ 0 & 0 & 1/\sqrt{3}\end{pmatrix}$$

Then $QR = A$, Q has orthonormal column vectors; so $Q^TQ = I_3$

c) With the QR factorization of A; $A^TAx = A^Tb$ is equivalent to $R^TQ^TQRx = R^TQ^Tb$ and $Q^TQ = I_3$. Now or $Rx = Q^Tb$ since R^T is nonsingular

$Q^Tb = \left(\frac{9}{5}, -\frac{17}{5\sqrt{6}}, \frac{4}{\sqrt{3}}\right)^T$. Solving $Rx = Q^Tb$ yields; $x_3 = 4$

$$\frac{\sqrt{6}\cdot\sqrt{6}\,x_2 + 4x_3}{5\sqrt{6}} = -\frac{17}{5\sqrt{6}} \rightarrow x_2 = -\frac{2}{3}x_3 - \frac{17}{6} = -\frac{16}{6} - \frac{17}{6} = -\frac{33}{6}$$

$$25x_1 + 13x_2 + 17x_3 = 9 \rightarrow x_1 = -\frac{13}{25}x_2 - \frac{17}{25}x_3 + \frac{9}{25} = \frac{13(33)}{150} - \frac{68}{25} + \frac{9}{25} = \frac{1}{2}$$

Solution of Least squares Pb $= \left(\frac{1}{2}, -\frac{33}{6}, 4\right)^T$