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Desensitizing control for a semilinear wave equation

Louis TEBOU
Department of Mathematics
Florida International University, Miami, USA

1. Problem formulation and statement of the main result

$\Omega =$ bounded domain in $\mathbb{R}^d$, $d \geq 1$,
$\Gamma =$ boundary of (Ω) class C^2, $\nu =$unit normal pointing into the exterior of Ω.
$\Gamma_0 = \{x \in \Gamma; (x - x^0) \cdot \nu \geq 0\}$.
$T > 0$, $Q = \Omega \times (0, T)$
$\omega =$ nonvoid open subset in Ω.

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuously differentiable function with

$$f(0) = 0, \quad \lim_{|s|\to\infty} \frac{f(s)}{s\sqrt{\log|s|}} = 0.$$

Let $\xi \in L^2(Q)$, and consider the wave equation:

$$\begin{cases} y_{tt} - \Delta y + f(y) = \xi + v\chi_\omega \text{ in } Q \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ y(0) = y^0 + \tau_0 \hat{y}^0; \quad y_t(0) = y^1 + \tau_1 \hat{y}^1 \text{ in } \Omega \end{cases} \tag{1}$$

where $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ are given, $(\hat{y}^0, \hat{y}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ have unit norm and are unknown, and τ_0 and τ_1 are small unknown real numbers.

We want to find a control v that desensitizes the functional Φ defined by

$$\Phi(y) = \frac{1}{2}\int_0^T \int_{\Gamma_0} |\frac{\partial y(\gamma,t)}{\partial \nu}|^2 \, d\gamma dt, \qquad (2)$$

that is to say, we are going to construct a control v satisfying for all $(\hat{y}^0, \hat{y}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ with unit norm:

$$\left.\frac{\partial \Phi(y)}{\partial \tau_0}\right|_{\tau_0=\tau_1=0} = 0 = \left.\frac{\partial \Phi(y)}{\partial \tau_1}\right|_{\tau_0=\tau_1=0}, \qquad (3)$$

History1: controllability of semilinear wave equation

a) Cirina (1969)

b) Chewning (1976)

Russell (1978),

c) Zuazua (1990-1991), HUM + Schauder fixed-point (linear growth); Zuazua (1993), HUM + Leray-Schauder (1d, superlinear growth allowed)

d) Lasiecka-Triggiani (1991), global inversion theorem (Lipschitz),

e) Li-Zhang (2000), Carleman estimates

f) Komornik-Loreti (2002), iterated log, improves Zuazua (1993)

g) Li-Rao (2003), quasi-linear hyperbolic equations,

h) Martinez-Vancostenoble (2003), 1d, arbitrarily short time,

i) Fu, Yong& Zhang (2007), Carleman estimates, hyperbolic equations,

j) Duyckaerts, Zhang & Zuazua (2008), improved Carleman estimates, allows

$$\limsup_{|s|\to\infty} \frac{f(s)}{s(\log|s|)^\alpha} = 0, \quad 0 \le \alpha < 3/2.$$

History2: control+incomplete data

a) J.L. Lions (1990), parabolic equations.

b) Bodart & Fabre (1995), de Teresa (1997), Bodart & *al* 2004, Fernandez-Cara & *al* (2005), ...

c) Desensitizing control concept ill-known for second order evolution equations,

d) Dáger (2006), desensitizing controls, one dimensional wave equation. The proof technique developed by Dáger critically relies on the fact that the one dimensional wave equation is time periodic, which is not the case in higher dimensions.

e) Tebou (2008), constructs desensitizing controls for the multidimensional wave equation using Carleman estimates, and suitable localizing techniques.

The functional that the control desensitizes in both Dáger and Tebou papers, as well as in almost all the papers dealing with parabolic equations is

$$\Phi(y) = \frac{1}{2} \int_0^T \int_{\mathcal{O}} |y(x,t)|^2 \, dxdt,$$

where $\mathcal{O}$ is another open subset of Ω.
Motivation. To examine what happens in the case of the new functional involving the normal derivative, and also to explore the case of a semilinear wave equation.

Our main result:

Theorem. *Let* $x_0 \in \mathbb{R}^d \setminus \bar{\Omega}$. *Set* $\Gamma_0 = \{x \in \Gamma; (x - x_0) \cdot \nu > 0\}$. *Assume that* ω *is a neighborhood of* Γ_0. *There exists a positive time* T_0^* *depending only on* Ω *and* ω *such that for every* $T > T_0^*$, *and for all* $y^0 \in H_0^1(\Omega)$ *and* $y^1 \in L^2(\Omega)$, *there exists a control* $v \in L^2(0, T; L^2(\omega))$ *that desensitizes the functional* Φ. *Moreover, there exists a positive constant* C *independent of the initial data such that:*

$$\begin{aligned} ||v||_{L^2(0,T;L^2(\omega))} &\leq C||\xi||_{L^2(Q)} \\ &+ C\left(||y^0||_{H_0^1(\Omega)} + ||y^1||_{L^2(\Omega)}\right). \end{aligned} \tag{4}$$

Remark 1. A precise structure of the constant T_0^* may be found in Duyckaerts-Zhang-Zuazua (2008).

2. Basic ideas for proving Theorem

The main ingredient for proving the existence of a desensitizing control is to reduce the problem to a controllability problem. To this end, consider the following cascade controlled wave equations:

$$\begin{cases} y_{0tt} - \Delta y_0 + f(y_0) = \xi + v\chi_\omega \text{ in } Q \\ y_0 = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1 \text{ in } \Omega \end{cases} \tag{5}$$

$$\begin{cases} q_{tt} - \Delta q + f'(y_0)q = \ \text{ in } Q \\ q = y_0\chi_{\Gamma_0} \text{ on } \Sigma = \partial\Omega \times (0,T) \\ q(T) = 0; \quad q_t(T) = 0 \text{ in } \Omega, \end{cases} \tag{6}$$

where y^0 and y^1 are the same as in Theorem. We have:

Proposition 1.*A control v desensitizes the functional* Φ if and only if the solution pair (y_0, q) of (5)-(6) satisfies:

$$q(0) = 0, \qquad q_t(0) = 0. \tag{7}$$

Remark 2. Proposition 1 reduces the proof of Theorem to showing that the nonlinear cascade system (5)-(6) is exactly controllable.

Set

$$g(s) = \begin{matrix} f(s)/s, \text{ if } s \neq 0 \\ f'(0), \text{ if } s = 0. \end{matrix}$$

Let $w \in L^\infty(0, T; L^2(\Omega))$. Set

$$a(x,t) = g(w(x,t)), \quad b(x,t) = f'(w(x,t)).$$

System (5)-(6) may be linearized as:

$$\begin{cases} y_{0tt} - \Delta y_0 + a y_0 = \xi + v\chi_\omega \text{ in } Q \\ y_0 = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1 \text{ in } \Omega \end{cases} \tag{8}$$

$$\begin{cases} q_{tt} - \Delta q + bq = 0 \text{ in } Q \\ q = \dfrac{\partial y_0}{\partial \nu} \chi_{\Gamma_0} \text{ on } \Sigma = \partial\Omega \times (0,T) \\ q(T) = 0; \quad q_t(T) = 0 \text{ in } \Omega, \end{cases} \tag{9}$$

Introduce the adjoint system to (5)-(6):

$$\begin{cases} p_{tt} - \Delta p + bp = 0 \text{ in } Q \\ p = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ p(0) = p^0; \quad p_t(0) = p^1 \text{ in } \Omega \end{cases} \tag{10}$$

$$\begin{cases} z_{tt} - \Delta z + az = 0 \text{ in } Q \\ z = \dfrac{\partial p}{\partial \nu} \chi_{\Gamma_0} \text{ on } \Sigma = \partial\Omega \times (0,T) \\ z(T) = 0; \quad z_t(T) = 0 \text{ in } \Omega. \end{cases} \tag{11}$$

For $(p^0, p^1) \in H_0^1(\Omega) \times L^2(\Omega)$, we have:

$$p \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)),$$
$$z \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega)).$$

For every $t \in [0,T]$, set

$$E(p;t) = \frac{1}{2}\left(|p_t(.,t)|^2_{L^2(\Omega)} + ||p(.,t)||^2_{H_0^1(\Omega)}\right).$$

Sketch of the proof of Theorem. The proof of Theorem essentially relies on

Proposition 2. *Let $\mathcal{O}$, ω, and T be given as in Theorem. There exists $C_1 > 0$ such that for all $(p^0, p^1) \in L^2(\Omega) \times H^{-1}(\Omega)$:*

$$E(p;0) \le C_1 \int_0^T \int_\omega |z_t(x,t)|^2 \, dxdt. \quad (10)$$

Introduce the functional

$$
\begin{aligned}
&\mathcal{J} : L^2(\Omega) \times H^{-1}(\Omega) \longrightarrow \mathbb{R} \\
&\qquad (p^0, p^1) \mapsto \mathcal{J}(p^0, p^1) \\
&\mathcal{J}(p^0, p^1) = \frac{1}{2} \int_0^T \int_\omega |z_t(x,t)|^2 \, dxdt \\
&+ \int_\Omega y^0(x) z_t(x,0) \, dx - \langle y^1, z(.,0) \rangle \\
&- \int_Q \xi(x,t) z(x,t) \, dxdt,
\end{aligned}
\tag{11}
$$

where $\langle .,. \rangle$ is the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. The functional $\mathcal{J}$ is strictly convex, and continuous. Further, $\mathcal{J}$ is coercive thanks to Proposition 2. Therefore, $\mathcal{J}$ has a unique minimizer (p^0, p^1), and if $\hat{z}$ is the corresponding solution of (9), then we have the Euler equation:

$$\begin{aligned} &\int_0^T \int_\omega \hat{z}_t(x,t) z_t(x,t)\, dxdt \\ &+ \int_\Omega y^0(x) z_t(x,0)\, dx - \langle y^1, z(.,0)\rangle \\ &- \int_Q \xi(x,t) z(x,t)\, dxdt = 0, \end{aligned} \tag{12}$$

for every z solution of (9). On the other hand, we have the duality identity:

$$\begin{aligned} &\langle p^1, q(.,0)\rangle - \int_\Omega p^0(x) q_t(x,0)\, dx \\ &= \langle y^1, z(.,0)\rangle - \int_\Omega y^0(x) z_t(x,0)\, dx \\ &+ \int_0^T \int_\omega v(x,t) z(x,t)\, dxdt \\ &+ \int_Q \xi(x,t) z(x,t)\, dxdt, \end{aligned} \tag{13}$$

Choosing the control
$v = \hat{z}_{tt} \in \left[H^1(0,T;L^2(\Omega))\right]'$ in (5), we derive from (12) and (13), for all$(p^0, p^1) \in L^2(\Omega) \times H^{-1}(\Omega)$:

$$\langle p^1, q(.,0)\rangle - \int_\Omega p^0(x) q_t(x,0)\, dx = 0, \quad (14)$$

hence (7). Therefore $v = \hat{z}_{tt}$ desensitizes the functional Φ, thanks to Proposition 1. It remains to show that v satisfies (4); this is easily done by setting $z = \hat{z}$ in (13), and noticing that:
$||\hat{z}_{tt}||_{[H^1(0,T;L^2(\Omega))]'} \leq ||\hat{z}_t||_{L^2(Q)}$. □

Sketch of the proof of Proposition 2. Let ω_0 and ω_1 be two neighborhoods of Γ_+ with $\omega_0 \subset \omega_1 \subset\subset \mathcal{O} \cap \omega$. Let $r \in C_0^\infty(0,T)$ denote the cut-off function defined in [DZZ, (2.33)], and set $\tilde{p} = rp$. Applying Theorem 2.4 of [DZZ] with $u = \tilde{p}$, we derive the existence of positive constants C, μ, and $\lambda_0 \geq 1$ such that for all $\lambda > \lambda_0$:

$$\begin{aligned} &\int_{T_0}^{T_0'} \int_\Omega |p(x,t)|^2 \, dxdt \\ &\leq C\lambda e^{-\mu\lambda} E(p;0) \\ &+ Ce^{C\lambda} \int_0^T r^2 \int_{\omega_0} |p(x,t)|^2 \, dxdt, \end{aligned} \tag{15}$$

where, here and in the sequel, the positive constant C may bear different values, and the constants T_0 and T_0' are given by [DZZ, (2.29)]. On the other hand one checks that:

$$E(p;0) \leq C \int_{T_0}^{T_0'} \int_{\Omega} |p(x,t)|^2 \, dxdt. \quad (16)$$

The combination of (15) and (16) yields for λ large enough:

$$E(p;0) \leq C \int_0^T r^2 \int_{\omega_0} |p(x,t)|^2 \, dxdt. \quad (17)$$

Introduce the function η, which satisfies: $\eta \in C^\infty(\bar{\Omega})$,
$0 \leq \eta \leq 1, \quad \eta = 1$ in $\omega_0, \quad \eta = 0,$ in $\Omega \setminus \omega_1, \quad \frac{|\nabla \eta|^2}{\eta} \in C^0(\bar{\Omega}), \quad \frac{|\Delta \eta|^2}{\eta} \in C^0(\bar{\Omega})$.

Set $\check{z} = r^2 \eta z$. One checks that for all $\varepsilon > 0$:

$$\begin{aligned}
&\int_{\omega_{1T}} r^2 \eta |p|^2 \, dxdt \\
&= -4 \int_{\omega_{1T}} r r' \eta z_t p \, dxdt \\
&+ \int_{\omega_{1T}} \left\{ -2\eta (r r'' + |r'|^2) z p \right\} \, dxdt \\
&+ \int_{\omega_{1T}} r^2 (2 \nabla \eta \cdot \nabla z + z \Delta \eta) p \, dxdt \\
&\leq \varepsilon \int_{\omega_{1T}} r^2 \eta |p|^2 \, dxdt \\
&+ C_\varepsilon \int_{\omega_{1T}} \left\{ |z|^2 + |z_t|^2 + r^2 |\nabla z|^2 \right\} \, dxdt.
\end{aligned} \tag{18}$$

Choosing $\varepsilon = 1/2$, we draw from (18):

$$\int_0^T r^2 \int_{\omega_0} |p|^2 \, dxdt \\ \leq C \int_{\omega_{1T}} \left\{ |z|^2 + |z_t|^2 + r^2 |\nabla z|^2 \right\} \, dxdt. \tag{19}$$

Introduce the function ζ, which satisfies $\zeta \in C^\infty(\bar{\Omega})$, $0 \leq \zeta \leq 1$, $\zeta = 1$ in ω_1, $\zeta = 0$, in $\Omega \setminus (\mathcal{O} \cap \omega)$. Set $\tilde{z} = r^2 \zeta z$. One shows that:

$$- \int_Q \tilde{z}_t z_t \, dxdt + \int_Q \nabla \tilde{z} \cdot \nabla z \, dxdt \\ = \int_Q \left\{ 4 r r' \zeta z_t z + 2\zeta (r r'' + |r'|^2) z^2 \right\} \, dxdt \\ - \int_Q r^2 (2 z \nabla \zeta \cdot \nabla z + z^2 \Delta \zeta + \zeta z p \chi_{\mathcal{O}}) \, dxdt. \tag{20}$$

It follows from (20), integrating by parts where needed, that for every $\delta > 0$:

$$\begin{aligned} &\int_{\omega_{1T}} r^2 |\nabla z|^2 \, dxdt \\ &\leq C_\delta \int_0^T \int_\omega \left\{ |z|^2 + |z_t|^2 \right\} \, dxdt \\ &+ 2\delta T E(p;0). \end{aligned} \tag{21}$$

Combining (17), (19), (21), choosing $\delta = 1/4CT$ (with C as the product of the constants in (17) and (19)), and using the Poincaré inequality (as $z(T) = 0$), we get the claimed estimate (10). □

Open problem. Does there exist a control that desensitizes the functional:

$$\Psi(y, y_t) = \int_0^T \int_{\mathcal{O}} \{|y|^2 + |\nabla G y_t|^2\} \, dxdt?$$

where G is the inverse of $-\Delta$ with Dirichlet boundary condition.