Problem Session 2 - key

1) Let A be $n\times n$ nonsingular. Show $\operatorname{adj}A$ is nonsingular and $(\operatorname{adj}A)^{-1} = \operatorname{adj}(A^{-1})$.

We have $A(\operatorname{adj}A) = \det(A)I_n$. If A is nonsingular, then

$$A^{-1}(A\operatorname{adj}A) = A^{-1}(\det(A)I_n) = \det(A)A^{-1}$$

Now $A^{-1}(\operatorname{Adj}A) = (A^{-1}A)\operatorname{adj}A = \operatorname{adj}A$, since $A^{-1}A = I_n$. So

$\det(A) \neq 0$ since A is nonsingular, and $\operatorname{adj}A = \det(A)A^{-1}$; so $\operatorname{adj}A$ is nonsingular, and

$$(\operatorname{adj}A)^{-1} = (\det(A)A^{-1})^{-1} = \frac{(A^{-1})^{-1}}{\det(A)}$$
$$= \det(A^{-1})(A^{-1})^{-1}$$
$$= \operatorname{adj}(A^{-1}).$$

2) Let $\beta_1, \beta_2, \ldots, \beta_n$ be scalars such that $\sum_{j=1}^{n}\beta_j w_j = 0_E$. When do we have $\beta_1 = 0, \beta_2 = 0, \ldots, \beta_n = 0$?

We have, by the definition of $w_1, w_2, \ldots, w_n$:

$$\sum_{j=1}^{n}\beta_j w_j = \sum_{j=1}^{n}\beta_j(u_j + v)$$
$$= \sum_{j=1}^{n}\beta_j u_j + \left(\sum_{j=1}^{n}\beta_j\right)v$$
$$= \sum_{j=1}^{n}\beta_j u_j + \sum_{j=1}^{n}\beta_j\sum_{k=1}^{n}\alpha_k u_k$$
$$= \sum_{k=1}^{n}\beta_k u_k + \sum_{k=1}^{n}\left(\alpha_k\left(\sum_j\beta_j\right)\right)u_k$$
$$= \sum_{k=1}^{n}\left(\beta_k + \alpha_k\sum_{j=1}^{n}\beta_j\right)u_k.$$

If $\sum\beta_j w_j = 0_E$, then $\sum_{k=1}^{n}\left(\beta_k + \alpha_k\sum_{j=1}^{n}\beta_j\right)u_k = 0_E$; so $\beta_k + \alpha_k\sum_{j=1}^{n}\beta_j = 0$, for all k,

since $u_1, u_2, \ldots, u_n$ are linearly independent.

Now $\left(\beta_k + \alpha_k\sum_{j=1}^{n}\beta_j = 0, \text{ for all } k\right) \Rightarrow \sum_{k=1}^{n}\left(\beta_k + \alpha_k\sum_{j=1}^{n}\beta_j\right) = 0$, taking the sum over k

$$\Rightarrow \sum_{k=1}^{n}\beta_k + \left(\sum_{k=1}^{n}\alpha_k\right)\left(\sum_{j=1}^{n}\beta_j\right) = 0$$

$$\Rightarrow \left(1 + \sum_{k=1}^{n}\alpha_k\right)\sum_{j=1}^{n}\beta_j = 0, \text{ since } \sum_{k=1}^{n}\beta_k = \sum_{j=1}^{n}\beta_j.$$

Thus $\sum_{k=1}^{n}\alpha_k \neq -1 \Rightarrow \sum_{j=1}^{n}\beta_j = 0$; so $\beta_k = 0$ for all k since $\beta_k + \alpha_k\sum_{j=1}^{n}\beta_j = 0$, for all k and $\sum_{j=1}^{n}\beta_j = 0$

Therefore $\sum_{k=1}^{n}\alpha_k \neq -1 \Rightarrow w_1, w_2, \ldots, w_n$ are linearly independent.

Suppose $w_1, w_2, \dots, w_n$ are linearly independent. Show that $\sum_{k=1}^{n} \alpha_k \neq -1$. It suffices to show that if $\sum_{k=1}^{n} \alpha_k = -1$, then $w_1, w_2, \dots, w_n$ are linearly dependent.

Assume $\sum_{k=1}^{n} \alpha_k = -1$. Show that $w_1, w_2, \dots, w_n$ are linearly dependent.

We have

$$w_j = u_j + v, \quad j = 1, 2, \dots, n$$

multiplying both sides by α_j and taking the sum over j, we find

$$\sum_{j=1}^{n} \alpha_j w_j = \sum_{j=1}^{n} \alpha_j u_j + \left(\sum_{j=1}^{n} \alpha_j\right) v = v - v = 0_E, \text{ as } \sum_{j=1}^{n} \alpha_j = -1.$$

So $\sum \alpha_j w_j = 0_E$, and $\alpha_1, \alpha_2, \dots, \alpha_n$ are not all zero, as $\sum \alpha_j = -1$; hence $w_1, w_2, \dots, w_n$ are linearly dependent.

3) $\mathbb{R}^3 = \mathrm{Span}(u_1, u_2, u_3)$. Do we have $\mathbb{R}^3 = \mathrm{Span}(u_1, u_1+u_2, u_1+u_2+u_3)$?

Let $x = (a, b, c)^T \in \mathbb{R}^3$. Can we find scalars $\alpha, \beta, \gamma \in \mathbb{R}^3$ such that

$x = \alpha u_1 + \beta(u_1+u_2) + \gamma(u_1+u_2+u_3)$. Now $x = a u_1 + b u_2 + c u_3$ as $\mathbb{R}^3 = \mathrm{Span}(u_1, u_2, u_3)$

If $x = \alpha u_1 + \beta(u_1+u_2) + \gamma(u_1+u_2+u_3) = a u_1 + b u_2 + c u_3$, then

$(\alpha+\beta+\gamma)u_1 + (\beta+\gamma)u_2 + \gamma u_3 = a u_1 + b u_2 + c u_3$; it suffices to

Choose $\gamma = c$, $\beta = b - c$, $\alpha = a - b$; hence $\mathbb{R}^3 = \mathrm{Span}(u_1, u_1+u_2, u_1+u_2+u_3)$.

4) a) Let U, V be subspaces of E. Show $\mathrm{Span}(U \cup V) = U + V$

$$\left.\begin{array}{l} U \subseteq U \cup V, \text{ so } U \subseteq \mathrm{span}(U \cup V) \\ V \subseteq U \cup V, \text{ so } V \subseteq \mathrm{span}(U \cup V) \end{array}\right\} \text{therefore } U + V \subseteq \mathrm{Span}(U \cup V) \text{ as}$$

$\mathrm{Span}(U \cup V)$ is a subspace of E; $\mathrm{Span}(U \cup V) + \mathrm{Span}(U \cup V) = \mathrm{Span}(U \cup V)$.

Similarly, $U \subseteq U + V$ and $V \subseteq U + V$; so $U \cup V \subseteq U + V$; hence $\mathrm{Span}(U \cup V) \subseteq \mathrm{Span}(U+V) = U + V$, as $U + V$ is a subspace of E.

Hence $\mathrm{Span}(U \cup V) = U + V$.

4) b) Let U, V, W be subspaces of E. Show

$$(U\cap V)+(U\cap W)\subseteq U\cap(V+W).$$

Let $x\in(U\cap V)+(U\cap W)$. Show that $x\in U\cap(V+W)$.

$x\in(U\cap V)+(U\cap W) \Rightarrow x=u+v,\ u\in U\cap V$ and $v\in U\cap W$

$\Rightarrow x\in U$, as $u+v\in U$, and $x\in V+W$, as $u\in V$ and $v\in W$

$\Rightarrow x\in U\cap(V+W)$.

Hence $(U\cap V)+(U\cap W)\subseteq U\cap(V+W)$.

c) Show $U+(V\cap W)\subseteq(U+V)\cap(U+W)$.

Let $x\in U+(V\cap W)$. Show that $x\in(U+V)\cap(U+W)$.

$x\in U+(V\cap W) \Rightarrow x=u+v,\ x\in u$ and $v\in V\cap W$

$\Rightarrow x=u+v\in U+V$, as $x\in U$ and $v\in V$, and
$x=u+v\in U+W$, as $x\in U$ and $v\in W$

$\Rightarrow x\in(U+V)\cap(U+W)$.

Hence $U+(V\cap W)\subseteq(U+V)\cap(U+W)$.

Note: b) shows that intersection does not distribute through addition, while c) shows that addition does not distribute through intersection. The next two questions show under which conditions those distributive laws hold.

d) From b), we already have: $(V\cap U)+(V\cap W)\subseteq V\cap(U+W)$. We shall now show: $V\cap(U+W)\subseteq(V\cap U)+(V\cap W)$ if $U\subseteq V$.

Let U, V, W be subspaces of E, while $U\subseteq V$. Let $x\in V\cap(U+W)$. Show $x\in(V\cap U)+(V\cap W)$.

$x\in V\cap(U+W) \Rightarrow x\in V$, and $x\in U+W$

$\Rightarrow x\in V$, and $x=u+w,\ u\in U,\ w\in W$

$\Rightarrow w=x-u\in V$, since $U\subseteq V$

$\Rightarrow x\in(U\cap V)+(V\cap W)$, since $U=U\cap V$.

Hence $V\cap(U+W)\subseteq(U\cap V)+(V\cap W)$. Therefore
$V\cap(U+W)=(V\cap U)+(V\cap W)$, if $U\subseteq V$.

e) From c) we already have $V+(U\cap W)\subseteq(V+U)\cap(V+W)$. We shall show $U\cap(V+W)\subseteq V+(U\cap W)$ if $V\subseteq U$. Note that $V\subseteq U\Rightarrow U+V=U$. Let $x\in U\cap(V+W)$. Show $x\in V+(U\cap W)$.

$x\in U\cap(V+W) \Rightarrow x\in U$ and $x=v+w,\ v\in V$ and $w\in W$.

$\Rightarrow w=x-v\in U$ since $V\subseteq U$.

$\Rightarrow x=v+w\in V+(U\cap W)$. So $U\cap(V+W)\subseteq V+(U\cap W)$.

Hence $V+(U\cap W)=U\cap(V+W)$ if $V\subseteq U$.

5) Set $A=(u_1, u_2, u_3, u_4)$. Let $b=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\in\mathbb{R}^4$. Does $AX=b$ have a unique solution?

$$\left(\begin{array}{cccc|c}1&2&0&1&a\\2&2&2&0&b\\3&2&4&-1&c\\4&6&4&2&d\end{array}\right)\xrightarrow[\substack{-3r_1+r_3\\-4r_1+r_4}]{-2r_1+r_2}\left(\begin{array}{cccc|c}1&2&0&1&a\\0&-2&2&-2&b-2a\\0&-4&4&-4&c-3a\\0&-2&4&-2&d-4a\end{array}\right)\xrightarrow[-r_2+r_4]{-2r_2+r_3}$$

$$\left(\begin{array}{cccc|c}1&2&0&1&a\\0&-2&2&-2&b-2a\\0&0&0&0&a-2b+c\\0&0&2&0&-2a-b+d\end{array}\right)$$

This linear system is inconsistent when $a-2b+c\neq 0$. So if $b=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\in\mathbb{R}^4$, and $a-2b+c\neq 0$, then $b\notin \operatorname{span}(u_1, u_2, u_3, u_4)$

So u_1, u_2, u_3, u_4 do not span $\mathbb{R}^4$. What about u_1, u_2, u_3, u_5.

Set $B=(u_1, u_2, u_3, u_5)$. Let $b=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$. Is $Ax=b$ consistent?

$$\left(\begin{array}{cccc|c}1&2&0&2&a\\2&2&2&3&b\\3&2&4&0&c\\4&6&4&1&d\end{array}\right)\xrightarrow[\substack{-3r_1+r_3\\-4r_1+r_4}]{-2r_1+r_2}\left(\begin{array}{cccc|c}1&2&0&2&a\\0&-2&2&-1&b-2a\\0&-4&4&-6&c-3a\\0&-2&4&-7&d-4a\end{array}\right)\xrightarrow[-r_2+r_4]{-2r_2+r_3}$$

$$\left(\begin{array}{cccc|c}1&2&0&2&a\\0&-2&2&-1&b-2a\\0&0&0&-4&a-2b+c\\0&0&2&-6&-2a-b+d\end{array}\right)\xrightarrow{r_3\rightleftarrows r_4}\left(\begin{array}{cccc|c}1&2&0&2&a\\0&-2&2&-1&b-2a\\0&0&2&-6&-2a-b+d\\0&0&0&-4&a-2b+c\end{array}\right)$$

System is consistent no matter what the values of a, b, c, d are; so $\mathbb{R}^4=\operatorname{span}(u_1, u_2, u_3, u_5)$.

6) Let α, β, γ in $\mathbb{R}$ such that $\alpha v_1+\beta v_2+\gamma v_3=0_E$. Do we have $\alpha=0, \beta=0, \gamma=0$?

$$\begin{aligned}\alpha v_1+\beta v_2+\gamma v_3&=\alpha(u_1-2u_2)+\beta(2u_2-3u_3)+\gamma(3u_3-4u_1)\\&=(\alpha-4\gamma)u_1+2(-\alpha+\beta)u_2+3(-\beta+\gamma)u_3\\&=0_E\rightarrow \alpha=4\gamma,\ \beta=\alpha,\ \gamma=\beta, \text{ as } u_1, u_2, u_3 \text{ linearly independent}\\&\rightarrow \alpha=4\alpha\rightarrow\alpha=0,\ \beta=0,\ \gamma=0;\end{aligned}$$

so v_1, v_2, v_3 are linearly independent.