Carleman inequalities for the wave equation with dynamic Wentzell boundary conditions.

Ciprian Gal & Louis Tebou

Florida International University, Miami, USA

KUMUNU 2017
Lincoln, NE

April 23, 2017

Overview

- Model equations and some earlier works

Overview

- Model equations and some earlier works
- First Carleman estimate.

Overview

- Model equations and some earlier works
- First Carleman estimate.
- An improved Carleman estimate.

Overview

- Model equations and some earlier works
- First Carleman estimate.
- An improved Carleman estimate.
- The Controllability problem.

Overview

- Model equations and some earlier works
- First Carleman estimate.
- An improved Carleman estimate.
- The Controllability problem.
- Perspectives

Model equations

Consider the following system

$$\begin{cases} \partial_t^2 u - d\Delta u = g, \text{ in } \Omega_T = \Omega \times (-T, T), \\ \partial_t^2 u_\Gamma + d\partial_\nu u - \delta \Delta_\Gamma u_\Gamma = g_\Gamma, \quad u = u_\Gamma \text{ on } \Gamma_T^1 = \Gamma^1 \times (-T, T), \\ u = 0 \text{ on } \Gamma_T^2 = \Gamma^2 \times (-T, T), \\ (u(-T), u_\Gamma(-T)) = (u_0, \widetilde{u}_0) \text{ in } \Omega \times \Gamma^1 \\ (\partial_t u(-T), \partial_t u_\Gamma(-T)) = (u_1, \widetilde{u}_1) \text{ in } \Omega \times \Gamma^1. \end{cases}$$

$T > 0$, Ω= bounded domain in $\mathbb{R}^N$, $N \geq 1$,
u denotes the amplitude of the wave in Ω, while u_Γ represents the amplitude of the wave on Γ^1, $g \in L^2(\Omega_T)$, $g_\Gamma \in L^2(\Gamma_T^1)$ are some source terms, and $\delta \geq 0$, $d > 0$ are constants.

For each $s \in \mathbb{N}$, set

$$H^s_{\Gamma_2} = \{u \in H^s(\Omega); u = 0 \text{ on } \Gamma_2\},$$
$$\mathcal{V}^s_\delta = \{(u, u_\Gamma) \in H^s_{\Gamma_2} \times H^1(\Gamma^1); u = u_\Gamma \text{ on } \Gamma^1\}.$$

It can be shown that the given dynamical system has a unique weak solution couple

$$(u, u_\Gamma) \in C([-T, T]\,; \mathcal{V}_\delta) \cap C^1([-T, T]\,; H),$$

where $\mathcal{V}_\delta = \mathcal{V}^1_\delta$ and $H = L^2(\Omega) \times L^2(\Gamma)$.

For each $s \in \mathbb{N}$, set

$$H^s_{\Gamma_2} = \{u \in H^s(\Omega); u = 0 \text{ on } \Gamma_2\},$$
$$\mathcal{V}^s_\delta = \{(u, u_\Gamma) \in H^s_{\Gamma_2} \times H^1(\Gamma^1); u = u_\Gamma \text{ on } \Gamma^1\}.$$

It can be shown that the given dynamical system has a unique weak solution couple

$$(u, u_\Gamma) \in C([-T, T]\,; \mathcal{V}_\delta) \cap C^1([-T, T]\,; H),$$

where $\mathcal{V}_\delta = \mathcal{V}^1_\delta$ and $H = L^2(\Omega) \times L^2(\Gamma)$.
Further, if Ω is, say, $C^{1,1}$, and the initial data are smooth enough, then

$$(u, u_\Gamma) \in C\left([-T, T]\,; \mathcal{V}^2_\delta\right) \cap C^1([-T, T]\,; \mathcal{V}_\delta).$$

Some earlier works

The model at hand is closely related to the model of waves with acoustic boundary conditions. Those models have attracted a lot of attention as far as well-posedness (Goldstein et *al*, Lasiecka et *al*, C. Gal,...), feedback control(M. Cavalcanti et *al*, Lasiecka et *al*, Muñoz Rivera et *al*,...), controllability (Avalos-Lasiecka, A. Heminna,...) are concerned.

Some earlier works

The model at hand is closely related to the model of waves with acoustic boundary conditions. Those models have attracted a lot of attention as far as well-posedness (Goldstein et *al*, Lasiecka et *al*, C. Gal,...), feedback control(M. Cavalcanti et *al*, Lasiecka et *al*, Muñoz Rivera et *al*,...), controllability (Avalos-Lasiecka, A. Heminna,...) are concerned.

In particular, the authors dealing with feedback control or controllability relies on multiplier techniques *à la* Lions-Komornik. But this approach, though very flexible, is inefficient when lower order terms are present in the equations; the Carleman estimates are the tools of choice in dealing with the presence of lower order terms.

The technique of Carleman estimate was introduced by Carleman in 1939 in the context of unique continuation for an elliptic equation in $\mathbb{R}^2$. This technique was later refined and generalized to general second order operators by the works of e.g. Hörmander, Nirenberg, Calderón, Kenig, Tataru, Ruiz in the context of unique continuation for the Cauchy problem, and Isakov, Klibanov et *al*, Imanuvilov, Yamamoto et *al* in the context of inverse problems.

The technique of Carleman estimate was introduced by Carleman in 1939 in the context of unique continuation for an elliptic equation in $\mathbb{R}^2$. This technique was later refined and generalized to general second order operators by the works of e.g. Hörmander, Nirenberg, Calderón, Kenig, Tataru, Ruiz in the context of unique continuation for the Cauchy problem, and Isakov, Klibanov et *al*, Imanuvilov, Yamamoto et *al* in the context of inverse problems.

In particular, contributions involving Carleman estimates for hyperbolic equations include works by e.g. Lasiecka-Triggiani et *al*, Zhang, Imanuvilov, Osses et *al*, Ervedoza et *al*,... They all deal with either Dirichlet or Neumann BCs.

Geometric condition and weight functions

Geometric condition on the domain:

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are closed with } \Gamma_1 \cap \Gamma_2 = \emptyset.$$

$$\exists\, x_0 \notin \overline{\Omega}, \text{ such that } \begin{cases} (x - x_0) \cdot \nu(x) \le 0, & \text{for all } x \in \Gamma^1 \\ (x - x_0) \cdot \nu(x) \ge 0, & \text{for all } x \in \Gamma^2. \end{cases}$$

Geometric condition and weight functions

Geometric condition on the domain:

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are closed with } \Gamma_1 \cap \Gamma_2 = \emptyset.$$

$$\exists\, x_0 \notin \overline{\Omega}, \text{ such that } \begin{cases} (x - x_0) \cdot \nu(x) \leq 0, & \text{for all } x \in \Gamma^1 \\ (x - x_0) \cdot \nu(x) \geq 0, & \text{for all } x \in \Gamma^2. \end{cases}$$

Weight functions: Assume that Ω satisfies (**GC**) for some $x_0 \notin \overline{\Omega}$. Let $\beta > 0$, and introduce the weight φ as follows: for $(x, t) \in \overline{\Omega} \times (-T, T)$, set

$$\psi(x, t) = |x - x_0|^2 - \beta t^2 + C_0, \quad \text{and for } \lambda > 0, \quad \varphi(x, t) = e^{\lambda \psi(x,t)},$$

where $C_0 > 0$ is chosen such that $\psi \geq 1$ in $\Omega \times (-T, T)$.

Main hypotheses

Let $T > 0$ be arbitrary. Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Assume that the boundary components Γ^1 and Γ^2 also satisfy the geometric condition **GC.** Let ψ and φ be the weight functions just defined. Let $x_0 \notin \overline{\Omega}$ be such that one of the following conditions hold:

$$\begin{cases} (i): \delta > 0,\ d \geq \delta \text{ and } \partial_\nu \psi < 0 \text{ on } \Gamma^1,\ \text{with } \beta \in (0, \delta)\,, \\ (ii): \delta = 0,\ d > 0 \text{ and } \partial_\nu \psi < 0 \text{ on } \Gamma^1,\ \text{with } \beta \in (0, d)\,, \\ (iii): \delta > 0,\ d < \delta \text{ and } \partial_\nu \psi < 0 \text{ on } \Gamma^1 \text{ and } \left(N_{x_0}\right),\ \text{with } \beta \in (0, d)\,, \end{cases}$$

where the condition (N_{x_0}) reads as follows:

$$4\delta\left(\delta - \beta\right) + 2d\left(\delta - d\right)\left(x - x_0\right) \cdot \nu \geq 0, \text{ for all } x \in \Gamma^1.$$

Theorem 1

Under the main hypotheses, there exist $\lambda_0 > 0$, $s_0 > 0$ and a positive constant M such that for all $s \geq s_0$ and $\lambda \geq \lambda_0$,

$$
\begin{aligned}
& s\lambda \int_{\Omega_T} \varphi e^{2s\varphi}(s^2\lambda^2\varphi^2 v^2 + |\partial_t v|^2 + |\nabla v|^2) dxdt + s^3\lambda^3 \int_{\Gamma_T^1} e^{2s\varphi}\varphi^3 v^2 dSdt \\
& + \lambda s \int_{\Gamma_T^1} \varphi \left(\partial_\nu v\right)^2 e^{2s\varphi} d\Sigma \leq M \int_{\Omega_T} e^{2s\varphi} |g|^2 dxdt + M \int_{\Gamma_T^1} e^{2s\varphi} |g_\Gamma|^2 d\Sigma \\
& + Ms\lambda \int_{\Gamma_T^2} \varphi e^{2s\varphi} |\partial_\nu v|^2 dSdt + Ms\lambda \int_{\Gamma_T^1} \left(|\partial_t v_\Gamma|^2 + |\nabla_\Gamma v_\Gamma|^2\right) \varphi e^{2s\varphi} dSdt.
\end{aligned}
$$

for every $(v, v_\Gamma) \in L^2(-T, T; \mathcal{V}_\delta)$ satisfying

$$
\partial_t^2 v - d\Delta v = g \in L^2(\Omega_T),\ \partial_t^2 v_\Gamma + d\partial_\nu v - \delta\Delta_\Gamma v_\Gamma = g_\Gamma \in L^2(\Gamma_T^1)
$$

and $v(\pm T) = 0$ in Ω and $v_\Gamma(\pm T) = 0$, on Γ^1.

Proof: key elements

- Work with sufficiently regular solutions so that we can use v for both v and v_Γ.

Proof: key elements

- Work with sufficiently regular solutions so that we can use v for both v and v_Γ.
- Introduce a new variable

$$w(x,t) = v(x,t)e^{s\varphi(x,t)}, \quad \text{for all } (x,t) \in \Omega \times (-T,T).$$

and define the operators P, P_Γ by

$$Pw = e^{s\varphi}\left(\partial_t^2 - d\Delta\right)(e^{-s\varphi}w) \text{ and}$$
$$P_\Gamma w = e^{s\varphi}\left(\partial_t^2 + d\partial_\nu - \delta\Delta_\Gamma\right)(e^{-s\varphi}w).$$

- Some elementary computations yield

$$\begin{aligned} Pw &= \partial_t^2 w - 2s\lambda\varphi(\partial_t w \partial_t \psi - d\nabla w \cdot \nabla\psi) \\ &\quad + s^2\lambda^2\varphi^2 w(|\partial_t\psi|^2 - d|\nabla\psi|^2) - d\Delta w \\ &\quad - s\lambda\varphi w(\partial_t^2\psi - d\Delta\psi) - s\lambda^2\varphi w(|\partial_t\psi|^2 - d|\nabla\psi|^2) \\ &= P_1 w + P_2 w + Rw, \end{aligned}$$

where P_1 is given by

$$P_1 w = \partial_t^2 w - d\Delta w + s^2\lambda^2\varphi^2 w(|\partial_t\psi|^2 - d|\nabla\psi|^2)$$

and

$$\begin{aligned} P_2 w &= (\alpha - 1)s\lambda\varphi w(\partial_t^2\psi - d\Delta\psi) - s\lambda^2\varphi w(|\partial_t\psi|^2 - d|\nabla\psi|^2) \\ &\quad - 2s\lambda\varphi(\partial_t w \partial_t\psi - d\nabla w \cdot \nabla\psi) \\ Rw &= -\alpha s\lambda\varphi w(\partial_t^2\psi - d\Delta\psi), \end{aligned}$$

where

$$\alpha \in \left(\frac{2\beta}{\beta + nd}, \frac{2d}{\beta + nd}\right).$$

- Analogously, on $\Gamma^1_T = \Gamma^1 \times (-T, T)$, we get

$$P_\Gamma w = Q_1 w + Q_2 w + R_\Gamma w,$$

where

$$Q_1 w = \partial_t^2 w + d\partial_\nu w - \delta\Delta_\Gamma w + s^2\lambda^2\varphi^2 w\left(|\partial_t\psi|^2 - \delta\,|\nabla_\Gamma\psi|^2\right)$$
$$Q_2 w = -s\lambda^2\varphi w(|\partial_t\psi|^2 - \delta|\nabla_\Gamma\psi|^2) - 2s\lambda\varphi(\partial_t w\partial_t\psi - \delta\nabla_\Gamma w\cdot\nabla_\Gamma\psi)$$
$$R_\Gamma w = -s\lambda\varphi w(\partial_t^2\psi + d\partial_\nu\psi - \delta\Delta_\Gamma\psi).$$

- We have

$$\int_{\Omega_T} \left(|P_1 w|^2 + |P_2 w|^2\right) \, dxdt + \; 2 \int_{\Omega_T} P_1 w P_2 w \, dxdt$$
$$+ \int_{\Gamma_T^1} \left(|Q_1 w|^2 + |Q_2 w|^2\right) \, dSdt + \; 2 \int_{\Gamma_T^1} Q_1 w Q_2 w \, dSdt$$
$$= \int_{\Omega_T} |Pw - Rw|^2 \, dxdt + \int_{\Gamma_T^1} |P_\Gamma w - R_\Gamma w|^2 \, dSdt.$$

- We have

$$\int_{\Omega_T} \left(|P_1 w|^2 + |P_2 w|^2\right) \, dxdt + \; 2 \int_{\Omega_T} P_1 w P_2 w \, dxdt$$
$$+ \int_{\Gamma_T^1} \left(|Q_1 w|^2 + |Q_2 w|^2\right) \, dSdt + \; 2 \int_{\Gamma_T^1} Q_1 w Q_2 w \, dSdt$$
$$= \int_{\Omega_T} |Pw - Rw|^2 \, dxdt + \int_{\Gamma_T^1} |P_\Gamma w - R_\Gamma w|^2 \, dSdt.$$

- Appropriately bound from below the mixed terms:

$$\int_{\Omega_T} P_1 w \, P_2 w \, dxdt \text{ and } \int_{\Gamma_T^1} Q_1 w Q_2 w \, dSdt$$

Some useful notations

We now state and prove a better Carleman estimate where the nullity of initial and final displacements of the system is dropped.
But, first, some additional notations.
Let $\varepsilon > 0$ be a constant close to one, and introduce the function $r \in C^2([-T, T])$ with $r(-T) = 0 = r(T)$, and $r \equiv 1$ on $[-\varepsilon T, \varepsilon T]$. Set

$$R = \max\{|x - x_0|; x \in \bar{\Omega}\}, \quad T_0 = \begin{cases} R \max(\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{\delta}}) \text{ if } \delta > 0 \\ \frac{R}{\sqrt{d}} \text{ if } \delta = 0. \end{cases}$$

Theorem 2.

Let $T > T_0$. Let ψ, φ, $\beta > 0$, $x_0 \notin \overline{\Omega}$, δ and d be given as in Theorem 1. Set $\lambda = \lambda_0$, where $\lambda_0 \geq 1$ is given by Theorem 1. There exist $s_1 > 0$ and a positive constant M such that for all $s \geq s_1$, one has

$$
\begin{aligned}
& s\int_{\Omega_T} e^{2s\varphi}(s^2v^2 + |\partial_t v|^2 + |\nabla v|^2)dxdt + s^2\int_{\Gamma_T^1} e^{2s\varphi}v^2 d\Sigma \\
& + s\int_{\Gamma_T^1}(|\partial_t v|^2 + \delta\,|\nabla_\Gamma v|^2 + r^2|\partial_\nu v|^2)e^{2s\varphi}d\Sigma \leq M\int_{\Omega_T} e^{2s\varphi}\,|\mathcal{P}v|^2\,dxdt \\
& + M\int_{\Gamma_T^1} e^{2s\varphi}\,|\mathcal{P}_\Gamma v_\Gamma|^2\,dxdt + Ms\int_{\Gamma_T^2} e^{2s\varphi}\,|\partial_\nu v|^2\,d\Sigma \\
& + Ms\int_{\Gamma_T^1} r^2(|\partial_t v_\Gamma|^2 + |\nabla_\Gamma v_\Gamma|^2)e^{2s\varphi}d\Sigma, \text{ for every } (v, v_\Gamma) \in L^2(-T, T; \mathcal{V}_\delta)
\end{aligned}
$$

with
$\mathcal{P}v = \partial_t^2 v - d\Delta v \in L^2(\Omega_T)$, $\mathcal{P}_\Gamma v_\Gamma = \partial_t^2 v_\Gamma + d\partial_\nu v - \delta\Delta_\Gamma v_\Gamma \in L^2(\Gamma_T^1)$.

key ideas for the proof of Theorem 2

- Use the cut-off function r to build $w = rv$.

key ideas for the proof of Theorem 2

- Use the cut-off function r to build $w = rv$.
- We have

$$\mathcal{P}w = r\mathcal{P}v + 2r'\partial_t v + r''v, \text{ and } \mathcal{P}_\Gamma w = r\mathcal{P}_\Gamma v + 2r'\partial_t v + r''v.$$

key ideas for the proof of Theorem 2

- Use the cut-off function r to build $w = rv$.
- We have

$$\mathcal{P}w = r\mathcal{P}v + 2r'\partial_t v + r''v, \text{ and } \mathcal{P}_\Gamma w = r\mathcal{P}_\Gamma v + 2r'\partial_t v + r''v.$$

- w satisfies all the requirements of Theorem 1

- Set $\lambda = \lambda_0$ to derive

$$
\begin{aligned}
& s \int_{\Omega_T} \varphi e^{2s\varphi} (|\partial_t w|^2 + |\nabla w|^2) dxdt + s^3 \int_{\Omega_T} \varphi^3 e^{2s\varphi} w^2 dxdt \\
& + s^3 \int_{\Gamma_T^1} \varphi^3 e^{2s\varphi} w^2 dSdt + s \int_{\Gamma_T^1} \varphi \left(\partial_\nu w\right)^2 e^{2s\varphi} dSdt \\
& \leq M \int_{\Omega_T} e^{2s\varphi} \left|\mathcal{P} w\right|^2 dxdt + M \int_{\Gamma_T^1} e^{2s\varphi} \left|\mathcal{P}_\Gamma w\right|^2 d\Sigma \\
& + Ms \int_{\Gamma_T^2} \varphi e^{2s\varphi} \left|\partial_\nu w\right|^2 d\Sigma + Ms \int_{\Gamma_T^1} \left(|\partial_t w|^2 + |\nabla_\Gamma w|^2\right) \varphi e^{2s\varphi} d\Sigma.
\end{aligned}
$$

- Replace *w* with *rv* to get

$$
\begin{aligned}
& s\int_{\Omega_T} r^2\varphi e^{2s\varphi}(|\partial_t v|^2+|\nabla v|^2)dxdt + s^3\int_{\Omega_T} r^2\varphi^3 e^{2s\varphi} v^2 dxdt \\
& + s\int_{\Gamma_T^1} r^2\varphi e^{2s\varphi}(s^2\varphi^2 v^2+|\partial_\nu v|^2)d\Sigma + \leq M\int_{\Omega_T} e^{2s\varphi}\,|\mathcal{P}v|^2\,dxdt \\
& + M\int_{\Gamma_T^1} e^{2s\varphi}\,|\mathcal{P}_\Gamma v|^2\,dxdt + Ms\int_{\Gamma_T^2}\varphi e^{2s\varphi}\,|\partial_\nu v|^2\,d\Sigma \\
& + Ms\int_{\Gamma_T^1} r^2\left(|\partial_t v|^2+|\nabla_\Gamma v|^2\right)\varphi e^{2s\varphi} dSdt + Ms\int_{\Omega_T}\varphi e^{2s\varphi}|v|^2 dxdt \\
& + Ms\int_{\Gamma_T^1}\varphi e^{2s\varphi}|v|^2 d\Sigma.
\end{aligned}
$$

Introduce the weighted energy

$$
\begin{aligned}
E_s(t) &= \frac{1}{2}\int_{\Omega} e^{2s\varphi(x,t)}\left(|\partial_t v(x,t)|^2 + d|\nabla v(x,t)|^2\right) dx \\
&\quad + \frac{1}{2}\int_{\Gamma^1} e^{2s\varphi(S,t)}\left(|\partial_t v(S,t)|^2 + \delta|\nabla_\Gamma v(S,t)|^2\right) dS, \quad -T \le t \le T.
\end{aligned}
$$

Introduce the weighted energy

$$E_s(t) = \frac{1}{2}\int_\Omega e^{2s\varphi(x,t)}(|\partial_t v(x,t)|^2 + d|\nabla v(x,t)|^2)\,dx \\ + \frac{1}{2}\int_{\Gamma^1} e^{2s\varphi(S,t)}(|\partial_t v(S,t)|^2 + \delta|\nabla_\Gamma v(S,t)|^2)\,dS, \quad -T \le t \le T.$$

The last estimate may be recast as

$$s\int_{-T}^{T} r^2 \left(E_s(t) + s^2\int_\Omega v^2\, e^{2s\varphi} dx\right) dt + s\int_{\Gamma_T^1} r^2 e^{2s\varphi}(s^2 v^2 + |\partial_\nu v|^2) d\Sigma$$

$$\le M\int_{\Omega_T} e^{2s\varphi}\,|\mathcal{P}v|^2\,dxdt + M\int_{\Gamma_T^1} e^{2s\varphi}\,|\mathcal{P}_\Gamma v|^2\,d\Sigma + Ms\int_{\Gamma_T^2} e^{2s\varphi}\,|\partial_\nu v|^2\,d\Sigma$$

$$+ Ms\int_{\Gamma_T^1} r^2\left(|\partial_t v|^2 + |\nabla_\Gamma v|^2\right) e^{2s\varphi} d\Sigma + Ms\int_{\Omega_T} e^{2s\varphi}|v|^2 dxdt$$

$$+ Ms\int_{\Gamma_T^1} e^{2s\varphi}|v|^2\Sigma.$$

- Get rid of r from LHS; this is where choosing T large enough is needed.

- Get rid of r from LHS; this is where choosing T large enough is needed.
  Take the derivative in t of $E_s(t)$, and use Cauchy-Schwarz, fine estimates on φ , large enough T and s to derive

$$
\begin{aligned}
& s\int_{-T}^{T} E_s(t)\,dt + s^3\int_{\Omega_T} r^2 e^{2s\varphi}|v|^2 dxdt + s^3\int_{\Gamma_T^1} r^2 e^{2s\varphi} v^2 d\Sigma \\
& \le Ms\int_{-T}^{T} r^2 E_s(t)\,dt + M\int_{\Omega_T} e^{2s\varphi}(\mathcal{P}v)^2\,dxdt + M\int_{\Gamma_T^1} e^{2s\varphi}(\mathcal{P}_\Gamma v)^2\,d\Sigma \\
& \le M\int_{\Omega_T} e^{2s\varphi}\,|\mathcal{P}v|^2\,dxdt + M\int_{\Gamma_T^1} e^{2s\varphi}\,|\mathcal{P}_\Gamma v|^2\,d\Sigma \\
& + Ms\int_{\Gamma_T^2} e^{2s\varphi}\,|\partial_\nu v|^2\,d\Sigma + Ms\int_{\Gamma_T^1}\left(|r\partial_t v|^2 + |\nabla_\Gamma v|^2\right) e^{2s\varphi} d\Sigma \\
& + Ms\int_{\Omega_T} e^{2s\varphi}|v|^2 dxdt + Ms\int_{\Gamma_T^1} e^{2s\varphi}|v|^2\Sigma.
\end{aligned}
$$

Lemma

Let $\varphi \in C^2(\bar{\Omega})$ with $\partial_\nu \varphi \leq -k_0$ on Γ^1, for some $k_0 > 0$. There exist positive constants s_0 and M such that for every $s \geq s_0$, and every $v \in H^1(\Omega)$ with $v = 0$ on Γ^2, we have the weighted Poincaré inequality

$$s^2 \int_\Omega e^{2s\varphi}|v|^2 dx + s \int_{\Gamma^1} e^{2s\varphi}|v|^2 dS \leq M \int_\Omega e^{2s\varphi}|\nabla v|^2 dx.$$

Use Lemma to get rid of the bad terms from the RHS to complete the proof.

Controllability preparatory estimates

Corollary 1

Under the hypotheses of Theorem 2, we have for every $\delta > 0$:

$$s\int_{\Omega_T} e^{2s\varphi}(|\partial_t v|^2 + d|\nabla v|^2)dxdt + s^3\int_{\Omega_T} e^{2s\varphi}|v|^2 dxdt$$
$$+ s\int_{\Gamma_T^1}(|\partial_t v|^2 + \delta\,|\nabla_\Gamma v|^2)e^{2s\varphi}d\Sigma + s\int_{\Gamma_T^1} r^2|\partial_\nu v|^2 e^{2s\varphi}d\Sigma$$
$$\leq M\int_{\Omega_T} e^{2s\varphi}\,|\mathcal{P}v|^2\,dxdt + M\int_{\Gamma_T^1} e^{2s\varphi}\,|\mathcal{P}_\Gamma v_\Gamma|^2\,d\Sigma + Ms\int_{\Gamma_T^2} e^{2s\varphi}\,|\partial_\nu v|^2\,d\Sigma$$
$$+ Ms\int_{\Gamma_T^1} r^2\,|\partial_t v_\Gamma|^2\,e^{2s\varphi}d\Sigma + Ms^3\int_{\Gamma_T^1} e^{2s\varphi}|v_\Gamma|^2 d\Sigma.$$

Controllability preparatory estimates

Corollary 2

Under the hypotheses of Theorem 2, we have for $\delta = 0$:

$$
\begin{aligned}
& s\int_{\Omega_T} e^{2s\varphi}(|\partial_t v|^2 + d|\nabla v|^2)dxdt + s^3\int_{\Omega_T} e^{2s\varphi}|v|^2 dxdt \\
& + s\int_{\Gamma_T^1} |\partial_t v|^2 e^{2s\varphi} d\Sigma + s\int_{\Gamma_T^1} r^2 |\partial_\nu v|^2 e^{2s\varphi} d\Sigma \\
& \leq M\int_{\Omega_T} e^{2s\varphi}|\mathcal{P}v|^2 dxdt + M\int_{\Gamma_T^1} e^{2s\varphi}|\mathcal{P}_\Gamma v|^2 d\Sigma + Ms\int_{\Gamma_T^2} e^{2s\varphi}|\partial_\nu v|^2 d\Sigma \\
& + Ms\int_{\Gamma_T^1} |\nabla_\Gamma v_\Gamma|^2 e^{2s\varphi} d\Sigma + Ms^3\int_{\Gamma_T^1} e^{2s\varphi}|v_\Gamma|^2 d\Sigma.
\end{aligned}
$$

Problem formulation: Case $\delta > 0$

Recall $H = L^2(\Omega) \times L^2(\Gamma^1)$, and denote by $\mathcal{V}_\delta'$ the topological dual of $\mathcal{V}_\delta$. Consider the boundary controllability problem: Given $(y^0, z^0) \in H$, $(y^1, z^1) \in \mathcal{V}_\delta'$, and functions $p_1 \in L^\infty(\Omega_T)$, $p_2,\ p_3 \in C^1(\bar{\Omega}_T)$, as well as $q_1 \in L^\infty(\Gamma_T^1)$, $q_2,\ q_3 \in C^1(\bar{\Gamma}_T^1)$, find a control $v_1 \in L^2(-T, T; L^2(\Gamma^1)) \oplus [H^1(-T, T; L^2(\Gamma^1))]'$, and a control $v_2 \in L^2(\Gamma_T^2))$ such that if the couple (y, z) solves the system

$$\begin{cases} \partial_t^2 y - d\Delta y + p_1 y + p_2 y_t + p_3 \cdot \nabla y = 0 & \text{in } \Omega_T, \\ \partial_t^2 z - \delta \Delta_\Gamma z + q_1 z + q_2 z_t + \delta q_3 \cdot \nabla_\Gamma z & \\ +d\partial_\nu y + (p_3 \cdot \nu) y = v_1, \quad y = z & \text{on } \Gamma_T^1, \\ y = v_2 & \text{on } \Gamma_T^2, \\ (y(-T), z(-T)) = (y^0, z^0), \quad (y_t(-T), z_t(-T)) = (y^1, z^1), & \text{in } \Omega \times \Gamma^1, \end{cases}$$

then one has

$$(y(T), z(T)) = (0, 0), \quad (y_t(T), z_t(T)) = (0, 0) \text{ in } \Omega \times \Gamma^1.$$

The solution of the controlled system is defined by transposition: Let $f \in L^2(\Omega_T)$ and $g \in L^2(\Gamma_T^1)$, and consider the backward system

$$\begin{cases} \partial_t^2 u - d\Delta u + p_1 u - p_{2,t} u - p_2 u_t - (\mathrm{div} p_3) u - p_3 \cdot \nabla u = f & \text{in } \Omega_T, \\ \partial_t^2 w - \delta \Delta_\Gamma w + q_1 w - q_{2,t} w - q_2 w_t - \delta(\mathrm{div}_\Gamma q_3) w - \delta q_3 \cdot \nabla_\Gamma w & \\ +d\partial_\nu u + (p_3 \cdot \nu) u = g, \quad u = w & \text{on } \Gamma_T^1, \\ u = 0 & \text{on } \Gamma_T^2, \\ (u(T), w(T)) = (0,0) \in \mathcal{V}_\delta, & \\ (u_t(T), w_t(T)) = (0,0) \in L^2(\Omega) \times L^2(\Gamma^1). & \end{cases}$$

It can be shown that the backward system has a unique weak solution

$$(u, w) \in C([-T, T]; \mathcal{V}_\delta) \cap C^1([-T, T]; H)$$

.

If v_1, v_2, f and g are given as above, the solution of the controlled system is then given by:

$$\begin{aligned}
&\int_{\Omega_T} f(x,t)y(x,t)\,dxdt + \int_{\Gamma_T^1} g(S,t)z(S,t)\,dSdt \\
&= \langle (y^1, z^1), (u(-T), w(-T))\rangle - \int_{\Omega} y^0(x)u_t(x,-T)\,dx \\
&- \int_{\Gamma^1} z^0(S)w_t(S,-T)\,dS + \int_{\Omega} p_2(x,-T)u(x,-T)y^0(x)\,dx \\
&+ \int_{\Gamma^1} q_2(S,-T)w(S,-T)z^0(S)\,dS + d\int_{\Gamma_T^2} v_2(S,t)\partial_\nu u(S,t)\,dSdt \\
&- \int_{\Gamma_T^1} v_1(S,t)w(S,t)\,dSdt,
\end{aligned}$$

where $\langle , \rangle$ stands for the duality product between $\mathcal{V}_\delta$ and its topological dual, and the last integral stands for the duality between $H^1(-T,T;L^2(\Gamma^1))$ and its topological dual.

It can be shown that the controlled system has a unique weak solution

$$(y,z) \in C([-T,T];H) \cap C^1([-T,T];\mathcal{V}_\delta').$$

Thanks to Lions' Hilbert uniqueness method (HUM) , solving this controllability problem amounts to establishing an inverse or observability inequality for the following adjoint system:

$$\begin{cases} \partial_t^2 u - d\Delta u + p_1 u - p_{2,t} u - p_2 u_t - (\mathrm{div} p_3) u - p_3 \cdot \nabla u = 0 & \text{in } \Omega_T, \\ \partial_t^2 w - \delta \Delta_\Gamma w + q_1 w - q_{2,t} w - q_2 w_t - \delta(\mathrm{div}_\Gamma q_3) w - \delta q_3 \cdot \nabla_\Gamma w & \\ +d\partial_\nu u + (p_3 \cdot \nu) u = 0, \quad u = w & \text{on } \Gamma_T^1, \\ u = 0 & \text{on } \Gamma_T^2, \\ (u(T), w(T)) = (u^0, w^0) \in \mathcal{V}_\delta, & \\ (u_t(T), w_t(T)) = (u^1, w^1) \in L^2(\Omega) \times L^2(\Gamma^1). & \end{cases}$$

Observability inequality

Introduce the energy function $E_{u,w}(t) = \frac{1}{2}\int_{\Omega}(|u_t(x,t)|^2 + d|\nabla u(x,t)|^2)\,dx + \frac{1}{2}\int_{\Gamma^1}(|w_t(S,t)|^2 + \delta|\nabla_\Gamma w(S,t)|^2)\,dS$.

Proposition

Let $T > T_0$. Let $(u^0, w^0) \in \mathcal{V}_\delta$, $(u^1, w^1) \in H$, and let the functions p_j and q_j, $j = 1,\ 2,\ 3$, be given as above. Assume that the couple (u, w) is the corresponding solution of the adjoint system. There exists a positive constant M that is independent of the data (u^0, u^1) and (w^0, w^1) such that the following observability inequality holds

$$E_{u,w}(-T) \leq M\int_{\Gamma^1_T}(r^2|w_t|^2 + |w|^2)\,d\Sigma + M\int_{\Gamma^2_T}|\partial_\nu u|^2\,d\Sigma.$$

Proof Sketch of Proposition

Proof Sketch of Proposition

This proof will be carried out in two steps. In the first step, we shall establish an energy estimate, while in the second one, we use that energy inequality and Corollary 1 to derive the claimed observability estimate.

Proof Sketch of Proposition

This proof will be carried out in two steps. In the first step, we shall establish an energy estimate, while in the second one, we use that energy inequality and Corollary 1 to derive the claimed observability estimate.

Step 1: An energy inequality. The idea is to prove the estimate for smooth solutions, then use a density argument. Multiplying the first equation in the adjoint system by u_t and using Green's formula yield

$$E'_{u,w}(t) = -\int_{\Omega_T} \{(p_1 - p_{2,t} - \mathrm{div}(p_3))u - p_2 u_t - p_3 \cdot \nabla u\} u_t \, dx$$
$$- \int_{\Gamma^1} \{(q_1 - q_{2,t} - \delta \mathrm{div}_\Gamma(q_3) + p_3 \cdot \nu) w - q_2 w_t - \delta q_3 \cdot \nabla_\Gamma w\} w_t \, dS.$$

Applying Cauchy-Schwarz and Poincaré inequalities, we easily derive the energy inequality

$$E_{u,w}(t) \leq E_{u,w}(s) e^{M|t-s|}, \quad \forall s, \; t \in [-T, T].$$

Similarly, one shows

Step 2: Derivation of the observability inequality. It follows from Corollary 1

$$
\begin{aligned}
& s\int_{\Omega_T} e^{2s\varphi}(|\partial_t u|^2 + d|\nabla u|^2)dxdt + s^3\int_{\Omega_T} e^{2s\varphi}|u|^2 dxdt \\
& + s\int_{\Gamma_T^1}(|\partial_t w|^2 + \delta\,|\nabla_\Gamma w|^2)e^{2s\varphi}d\Sigma \\
& \leq M\int_{\Omega_T} e^{2s\varphi}\,|\mathcal{P}u|^2\,dxdt + M\int_{\Gamma_T^1} e^{2s\varphi}\,|\mathcal{P}_\Gamma w|^2\,d\Sigma + Ms\int_{\Gamma_T^2} e^{2s\varphi}\,|\partial_\nu u|^2\,d\Sigma \\
& + Ms\int_{\Gamma_T^1} r^2\,|\partial_t w|^2\,e^{2s\varphi}d\Sigma + Ms^3\int_{\Gamma_T^1} e^{2s\varphi}|w|^2 d\Sigma.
\end{aligned}
$$

Some elementary algebra yields

$$\int_{\Omega_T} e^{2s\varphi}\, |\mathcal{P}u|^2 \, dxdt$$

$$\leq M \int_{\Omega_T} e^{2s\varphi}(|\partial_t u|^2 + d|\nabla u|^2) dxdt + M \int_{\Omega_T} e^{2s\varphi}|u|^2 dxdt,$$

Some elementary algebra yields

$$\int_{\Omega_T} e^{2s\varphi} \left|\mathcal{P}u\right|^2 dxdt$$
$$\leq M \int_{\Omega_T} e^{2s\varphi}(|\partial_t u|^2 + d|\nabla u|^2)dxdt + M \int_{\Omega_T} e^{2s\varphi}|u|^2 dxdt,$$

and

$$\int_{\Gamma_T^1} e^{2s\varphi} \left|\mathcal{P}_\Gamma w\right|^2 dSdt$$
$$\leq M \int_{\Gamma_T^1} e^{2s\varphi}(|\partial_t w|^2 + \delta|\nabla_\Gamma w|^2)dSdt + M \int_{\Gamma_T^1} e^{2s\varphi}|w|^2 dSdt.$$

Combining those inequalities with the last Carleman estimate, and choosing a large enough s, we derive

$$s\int_{\Omega_T} e^{2s\varphi}(|\partial_t u|^2 + d|\nabla u|^2)dxdt + s\int_{\Gamma_T^1}(|\partial_t w|^2 + \delta\,|\nabla_\Gamma w|^2)e^{2s\varphi}d\Sigma$$
$$\leq Ms\int_{\Gamma_T^2} e^{2s\varphi}\,|\partial_\nu u|^2\,d\Sigma + Ms\int_{\Gamma_T^1} r^2\,|\partial_t w|^2\,e^{2s\varphi}d\Sigma + Ms^3\int_{\Gamma_T^1} e^{2s\varphi}|w|^2 d\Sigma.$$

Combining those inequalities with the last Carleman estimate, and choosing a large enough s, we derive

$$s\int_{\Omega_T} e^{2s\varphi}(|\partial_t u|^2 + d|\nabla u|^2)dxdt + s\int_{\Gamma_T^1}(|\partial_t w|^2 + \delta\,|\nabla_\Gamma w|^2)e^{2s\varphi}d\Sigma$$
$$\leq Ms\int_{\Gamma_T^2} e^{2s\varphi}\,|\partial_\nu u|^2\,d\Sigma + Ms\int_{\Gamma_T^1} r^2\,|\partial_t w|^2\,e^{2s\varphi}d\Sigma + Ms^3\int_{\Gamma_T^1} e^{2s\varphi}|w|^2 d\Sigma.$$

Given the definition of the energy $E_{u,w}$, one deduces from that estimate

$$\int_{-T}^{T} E_{u,w}(t)dt \leq M\int_{\Gamma_T^2}|\partial_\nu u|^2\,d\Sigma + M\int_{\Gamma_T^1} r^2\,|\partial_t w|^2\,d\Sigma + M\int_{\Gamma_T^1}|w|^2 d\Sigma.$$

Combining those inequalities with the last Carleman estimate, and choosing a large enough s, we derive

$$s \int_{\Omega_T} e^{2s\varphi}(|\partial_t u|^2 + d|\nabla u|^2)dxdt + s\int_{\Gamma_T^1}(|\partial_t w|^2 + \delta\,|\nabla_\Gamma w|^2)e^{2s\varphi}d\Sigma$$

$$\leq Ms\int_{\Gamma_T^2} e^{2s\varphi}\,|\partial_\nu u|^2\,d\Sigma + Ms\int_{\Gamma_T^1} r^2\,|\partial_t w|^2\,e^{2s\varphi}d\Sigma + Ms^3\int_{\Gamma_T^1} e^{2s\varphi}|w|^2 d\Sigma.$$

Given the definition of the energy $E_{u,w}$, one deduces from that estimate

$$\int_{-T}^{T} E_{u,w}(t)dt \leq M\int_{\Gamma_T^2}|\partial_\nu u|^2\,d\Sigma + M\int_{\Gamma_T^1} r^2\,|\partial_t w|^2\,d\Sigma + M\int_{\Gamma_T^1}|w|^2 d\Sigma.$$

The energy inequality then yield the claimed observability estimate.

Controllability Theorem

Theorem 3

Let $T > T_0$. Let $(y^0, z^0) \in H$, $(y^1, z^1) \in \mathcal{V}'_\delta$, and let the functions p_j and q_j, $j = 1,\ 2,\ 3$, be given as above. Then for every $\delta > 0$, there exists a unique control couple (v_1, v_2) of minimal norm in $(L^2(\Gamma^1_T) \oplus [H^1(-T, T; L^2(\Gamma^1))]') \times L^2(\Gamma^2_T)$ such that the states couple (y, z) of the controlled system comes to rest in time T.

Proof Sketch of Theorem 3

Introduce the following functional

$$
\begin{aligned}
\mathcal{J}: \quad & \mathcal{V}_\delta \times H \longrightarrow \mathbb{R} \\
& ((u^0, w^0), (u^1, w^1)) \mapsto \frac{1}{2} \int_{\Gamma_T^1} (r^2 |w_t|^2 + |w|^2)\, d\Sigma + \frac{d}{2} \int_{\Gamma_T^2} |\partial_\nu u|^2\, d\Sigma \\
& - \langle (y^1, z^1), (u(-T), w(-T)) \rangle + \int_\Omega y^0(x) u_t(x, -T)\, dx \\
& + \int_{\Gamma^1} z^0(S) w_t(S, -T)\, dS - \int_\Omega p_2(x, -T) u(x, -T) y^0(x)\, dx \\
& - \int_{\Gamma^1} q_2(S, -T) w(S, -T) z^0(S)\, dS,
\end{aligned}
$$

where the couple (u, w) is the corresponding solution of the adjoint system, and $\langle , \rangle$ stands for the duality product between $\mathcal{V}_\delta$ and its topological dual.

It can be checked that the quadratic functional $\mathcal{J}$ is continuous and strictly convex. Further, $\mathcal{J}$ is coercive; indeed, thanks to Proposition, there exist positive constants M and C_0 such that for all $((u^0, w^0), (u^1, w^1))$ in $\mathcal{V}_\delta \times H$, one has

$$\begin{aligned}\mathcal{J}((u^0, w^0), (u^1, w^1)) \geq & M||((u^0, w^0), (u^1, w^1))||^2_{\mathcal{V}_\delta \times H} \\ & - C_0||((u^0, w^0), (u^1, w^1))||_{\mathcal{V}_\delta \times H}.\end{aligned}$$

It can be checked that the quadratic functional $\mathcal{J}$ is continuous and strictly convex. Further, $\mathcal{J}$ is coercive; indeed, thanks to Proposition, there exist positive constants M and C_0 such that for all $((u^0, w^0), (u^1, w^1))$ in $\mathcal{V}_\delta \times H$, one has

$$\begin{aligned}\mathcal{J}((u^0, w^0), (u^1, w^1)) \geq & M\|((u^0, w^0), (u^1, w^1))\|^2_{\mathcal{V}_\delta \times H} \\ & - C_0\|((u^0, w^0), (u^1, w^1))\|_{\mathcal{V}_\delta \times H}.\end{aligned}$$

The last inequality enables one to check

$$\frac{\mathcal{J}((u^0, w^0), (u^1, w^1))}{\|((u^0, w^0), (u^1, w^1))\|_{\mathcal{V}_\delta \times H}} \longrightarrow \infty \text{ as } \|((u^0, w^0), (u^1, w^1))\|_{\mathcal{V}_\delta \times H} \longrightarrow \infty.$$

Therefore, $\mathcal{J}$ has a unique minimizer $(\hat{u}^0, \hat{u}^1, \hat{w}^0, \hat{w}^1)$ which solves the Euler equation

$$
\begin{aligned}
&\int_{\Gamma_T^1} (r^2 \hat{w}_t w_t + \hat{w} w)\, dSdt + d \int_{\Gamma_T^2} \partial_\nu \hat{u} \partial_\nu u \, dSdt \\
&-\langle (y^1, z^1), (u(-T), w(-T)) \rangle + \int_\Omega y^0(x) u_t(x, -T)\, dx \\
&- \int_\Omega p_2(x, -T) u(x, -T) y^0(x)\, dx + \int_{\Gamma^1} z^0(S) w_t(S, -T)\, dS \\
&- \int_{\Gamma^1} q_2(S, -T) w(S, -T) z^0(S)\, dS = 0,
\end{aligned}
$$

for every solution (u, w) of the adjoint system, and

Therefore, $\mathcal{J}$ has a unique minimizer $(\hat{u}^0, \hat{u}^1, \hat{w}^0, \hat{w}^1)$ which solves the Euler equation

$$\begin{aligned}
&\int_{\Gamma_T^1} (r^2 \hat{w}_t w_t + \hat{w} w)\, dSdt + d \int_{\Gamma_T^2} \partial_\nu \hat{u} \partial_\nu u \, dSdt \\
&-\langle (y^1, z^1), (u(-T), w(-T)) \rangle + \int_\Omega y^0(x) u_t(x, -T)\, dx \\
&-\int_\Omega p_2(x, -T) u(x, -T) y^0(x)\, dx + \int_{\Gamma^1} z^0(S) w_t(S, -T)\, dS \\
&-\int_{\Gamma^1} q_2(S, -T) w(S, -T) z^0(S)\, dS = 0,
\end{aligned}$$

for every solution (u, w) of the adjoint system, and

where the couple $(\hat{u}, \hat{w})$ is the solution of the adjoint system corresponding to the minimizer.

Therefore, $\mathcal{J}$ has a unique minimizer $(\hat{u}^0, \hat{u}^1, \hat{w}^0, \hat{w}^1)$ which solves the Euler equation

$$\begin{aligned}
&\int_{\Gamma_T^1} (r^2 \hat{w}_t w_t + \hat{w} w)\, dSdt + d \int_{\Gamma_T^2} \partial_\nu \hat{u} \partial_\nu u \, dSdt \\
&- \langle (y^1, z^1), (u(-T), w(-T)) \rangle + \int_\Omega y^0(x) u_t(x, -T)\, dx \\
&- \int_\Omega p_2(x, -T) u(x, -T) y^0(x)\, dx + \int_{\Gamma^1} z^0(S) w_t(S, -T)\, dS \\
&- \int_{\Gamma^1} q_2(S, -T) w(S, -T) z^0(S)\, dS = 0,
\end{aligned}$$

for every solution (u, w) of the adjoint system, and

where the couple $(\hat{u}, \hat{w})$ is the solution of the adjoint system corresponding to the minimizer.

One checks that the optimal controls are given by: $v_1 = (r^2 \hat{w}_t)_t - \hat{w}$ and $v_2 = \partial_\nu \hat{u}$.

1. It is obvious that the control region that we are using is not optimal. It's enough to assume that the control region Γ_c satisfies the Bardos-Lebeau-Rauch geometric control condition (GCC):*there exists $T > 0$ such that every ray of geometric optics enters Γ_c in a time less than T. This is yet to be done.*

1. It is obvious that the control region that we are using is not optimal. It's enough to assume that the control region Γ_c satisfies the Bardos-Lebeau-Rauch geometric control condition (GCC):*there exists $T > 0$ such that every ray of geometric optics enters Γ_c in a time less than T. This is yet to be done.*
2. Assuming now that we have the Wentzell boundary conditions on the whole boundary of Ω, what kind of controllability result can we get by applying the control on an arbitrarily small open nonempty portion of the boundary? Probably, only approximate controllability, but it would be interesting to use Carleman estimates to derive appropriate observability inequalites. This type of estimates is now well understood for Dirichlet, Neumann, and Robin boundary conditions thanks to works by Lebeau, Burcq, Fu,...

1. It is obvious that the control region that we are using is not optimal. It's enough to assume that the control region Γ_c satisfies the Bardos-Lebeau-Rauch geometric control condition (GCC):*there exists $T > 0$ such that every ray of geometric optics enters Γ_c in a time less than T. This is yet to be done.*
2. Assuming now that we have the Wentzell boundary conditions on the whole boundary of Ω, what kind of controllability result can we get by applying the control on an arbitrarily small open nonempty portion of the boundary? Probably, only approximate controllability, but it would be interesting to use Carleman estimates to derive appropriate observability inequalites. This type of estimates is now well understood for Dirichlet, Neumann, and Robin boundary conditions thanks to works by Lebeau, Burcq, Fu,...
3. One could also think about using a control that acts on a nonempty open subset of Ω.

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!