Stabilization of the wave equation with localized Kelvin-Voigt damping

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October 11-12, 2014

Overview

- Problem formulation

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- Well-posedness and strong stability

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- Polynomial stability

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- Exponential stability

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- Polynomial stability
- Exponential stability
- Some extensions and open problems

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$$\begin{aligned} & y_{tt} - \Delta y - div(a\nabla y_t) = 0 \text{ in } \Omega \times (0,\infty) \\ & y = 0 \text{ on } \Sigma = \Gamma \times (0,\infty), \quad y(0) = y^0, \quad y_t(0) = y^1 \text{ in } \Omega, \end{aligned}$$

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Ω= bounded domain in $\mathbb{R}^N$, $N \geq 1$,
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- The damping coefficient is nonnegative, bounded measurable, and is positive in a nonempty open subset ω of Ω,
- the system may be viewed as a model of interaction between an elastic material (portion of Ω where $a \equiv 0$), and a viscoelastic material (portion of Ω where $a > 0$).

Remark

If $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, then the system is well-posed in $H_0^1(\Omega) \times L^2(\Omega)$. Introduce the energy

$$E(t) = \frac{1}{2} \int_\Omega \{|y_t(x,t)|^2 + |\nabla y(x,t)|^2\}\, dx, \quad \forall t \geq 0.$$

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We have the dissipation law:

$$\frac{dE}{dt}(t) = -\int_\Omega a(x)|\nabla y_t(x,t)|^2\, dx \text{ a.e. } t > 0.$$

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Question 1: Does the energy approach zero?

Question 2: When the energy does go to zero, how fast is its decay, and under what conditions?

Introduce the Hilbert space over the field $\mathbb{C}$ of complex numbers $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$, equipped with the norm

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Setting $Z = \begin{pmatrix} y \\ y' \end{pmatrix}$, the system may be recast as:

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the unbounded operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & div(a\nabla .) \end{pmatrix}$$

with $D(\mathcal{A}) = \left\{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega); \Delta u + div(a\nabla v) \in L^2(\Omega)\right\}$.

Now if $(y^0, y^1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ then it can be shown that the unique solution of the system satisfies

$$y \in \mathcal{C}([0,\infty); H_0^1(\Omega)) \cap \mathcal{C}^1([0,\infty); H_0^1(\Omega)).$$

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This is what makes the stabilization problem at hand trickier than the case of a viscous damping ay_t, or more generally $ag(y_t)$ for an appropriate nonlinear function g.

Theorem 1 [Liu-Rao, 2006]

Suppose that ω is an arbitrary nonempty open set in Ω. Let the damping coefficient *a* be nonnegative, bounded measurable, and positive in ω. The operator $\mathcal{A}$ generates a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on $\mathcal{H}$, which is strongly stable:

$$\lim_{t\to\infty} ||S(t)Z^0||_{\mathcal{H}} = 0, \quad \forall Z^0 \in \mathcal{H}.$$

For the sequel we need the geometric constraint (GC) on the subset ω where the dissipation is effective.

(GC). There exist open sets $\Omega_j \subset \Omega$ with piecewise smooth boundary $\partial\Omega_j$, and points $x_0^j \in \mathbb{R}^N$, $j = 1,\ 2, ..., J$, such that $\Omega_i \cap \Omega_j = \emptyset$, for any $1 \leq i < j \leq J$, and:

$$\Omega \cap \mathcal{N}_\delta \left[\left(\bigcup_{j=1}^{J} \Gamma_j \right) \bigcup \left(\Omega \setminus \bigcup_{j=1}^{J} \Omega_j \right) \right] \subset \omega,$$

for some $\delta > 0$, where $\mathcal{N}_\delta(S) = \bigcup_{x \in S} \{y \in \mathbb{R}^N; |x - y| < \delta\},$ for $S \subset \mathbb{R}^N$, $\Gamma_j = \left\{ x \in \partial\Omega_j; (x - x_0^j) \cdot \nu^j(x) > 0 \right\}$, ν^j being the unit normal vector pointing into the exterior of Ω_j.

Theorem 2

Suppose that ω satisfies the geometric condition (GC). Let the damping coefficient a be nonnegative, bounded measurable, with $a(x) \geq a_0$ a.e. in ω, for some constant $a_0 > 0$. There exists a positive constant C such that the semigroup $(S(t))_{t\geq 0}$ satisfies:

$$||S(t)Z^0||_{\mathcal{H}} \leq \frac{C||Z^0||_{D(\mathcal{A})}}{\sqrt{1+t}}, \quad \forall Z^0 \in D(\mathcal{A}), \quad \forall t \geq 0.$$

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Remark. The polynomial decay estimate in Theorem 2 is in sharp contrast with what happens in the case of a viscous damping of the form ay_t or $ag(y_t)$ for a nondecreasing globally Lipschitz nonlinearity g; in fact, when (GC) holds, the geometric control condition of Bardos-Lebeau-Rauch (every ray of geometric optics intersects ω in a finite time T_0) is met, and exponential decay of the energy should be expected; this is by now well known:

- in the viscous damping framework thanks to works by Chen and collaborators, Dafermos, Haraux, Lasiecka and collaborators, Lebeau, Nakao, Rauch-Taylor, Zuazua, ...

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However, it was shown in the one-dimensional setting by Liu-Liu (1998) that exponential decay of the energy fails if the coefficient a is discontinuous along the interface; this should be the case in the multidimensional setting, but more work is needed.

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- Apply a theorem of Borichev-Tomilov on polynomial decay of bounded semigroups.

We shall prove that there exists a constant $C > 0$ such that for every $U = (f, g) \in \mathcal{H}$, the element $Z = (ib - \mathcal{A})^{-1}U = (u, v)$ in $D(\mathcal{A})$ satisfies:

$$||Z||_{\mathcal{H}} \leq Cb^2||U||_{\mathcal{H}}, \quad \forall b \in \mathbb{R},\ |b| \geq 1 \tag{1}$$

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may be recast as

$$\begin{aligned} &ibu - v = f \text{ in } \Omega \\ &ibv - \Delta u - \operatorname{div}(a(x)\nabla v) = g \text{ in } \Omega \\ &u = 0, \quad v = 0 \text{ on } \Gamma. \end{aligned} \tag{3}$$

Introduce the new function $u_1 = u - w$, where $w = G(\mathrm{div}(a\nabla v))$, with $G =$inverse of $-\Delta$ with Dirichlet BCs. One notes $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, and

$$||w||_{H_0^1(\Omega)} \leq \sqrt{|a|_\infty ||U||_{\mathcal{H}} ||Z||_{\mathcal{H}}}, \quad ||u_1||_{H_0^1(\Omega)} \leq ||Z||_{\mathcal{H}} + \sqrt{|a|_\infty ||U||_{\mathcal{H}} ||Z||_{\mathcal{H}}}.$$

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The second equation in (3) becomes

$$ibv - \Delta u_1 = g \text{ in } \Omega,$$

from which one derives

$$|b| ||v||_{H^{-1}(\Omega)} \leq ||u_1||_{H_0^1(\Omega)} + C|g|_2.$$

Let $J \geq 1$ be a an integer. For each $j = 1, 2, ..., J$, set $m^j(x) = x - x_0^j$. Let $0 < \delta_0 < \delta_1 < \delta$, where δ is the same as in the geometric condition stated above. Set

$$S = \left(\bigcup_{j=1}^{J} \Gamma_j\right) \bigcup \left(\Omega \setminus \bigcup_{j=1}^{J} \Omega_j\right),$$
$$Q_0 = \mathcal{N}_{\delta_0}(S), \quad Q_1 = \mathcal{N}_{\delta_1}(S), \quad \omega_1 = \Omega \cap Q_1,$$

and for each j, let φ_j be a function satisfying

$$\varphi_j \in W^{1,\infty}(\Omega), \quad 0 \leq \varphi_j \leq 1, \quad \varphi_j = 1 \text{ in } \bar{\Omega}_j \setminus Q_1, \quad \varphi_j = 0 \text{ in } \Omega \cap Q_0.$$

The usual multiplier technique leads to the estimate

$$||Z||_{\mathcal{H}}^2 \leq C||U||_{\mathcal{H}}^2 + C|b| \left| \sum_{j=1}^{J} \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{w} \, dx \right|. \tag{4}$$

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which combined with (4) yields the sought after estimate:

$$||Z||_{\mathcal{H}} \leq Cb^2 ||U||_{\mathcal{H}}.$$

Theorem 3

Suppose that ω satisfies the geometric condition (GC). As for the damping coefficient a, assume

$$a \in W^{1,\infty}(\Omega) \text{ with } |\nabla a(x)|^2 \leq M_0 a(x), \text{ a.e. in } \Omega,$$
$$a(x) \geq a_0 > 0 \text{ a.e. in } \omega_1,$$

for some positive constants M_0 and a_0.
The semigroup $(S(t))_{t \geq 0}$ is exponentially stable; more precisely, there exist positive constants M and λ with

$$||S(t)Z^0||_{\mathcal{H}} \leq M \exp(-\lambda t)||Z^0||_{\mathcal{H}}, \quad \forall Z^0 \in \mathcal{H}.$$

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Remark. In Liu-Rao (2006), the feedback control region ω is a neighborhood of the whole boundary, and the damping coefficient a should further satisfy $\Delta a \in L^\infty(\Omega)$.

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- Apply a theorem due to Prüss or Huang on exponential decay of bounded semigroups.

We shall prove that there exists a constant $C > 0$ such that for every $U = (f, g) \in \mathcal{H}$, the element $Z = (ib - \mathcal{A})^{-1}U = (u, v)$ in $D(\mathcal{A})$ satisfies:

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Thanks to the proof sketch of Theorem 2, we already have:

$$||Z||^2_{\mathcal{H}} \leq C||U||^2_{\mathcal{H}} + C|b| \left| \sum_{j=1}^{J} \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{w}\, dx \right|. \tag{6}$$

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With the smoothness and structural conditions on the coefficient a, it can be shown that, on the one hand

$$C|b| \left| \sum_{j=1}^{J} \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{w}\, dx \right| \leq C|b||\sqrt{a}v|_2(||U||^{\frac{1}{2}}_{\mathcal{H}}||Z||^{\frac{1}{2}}_{\mathcal{H}} + ||Z||_{\mathcal{H}}), \tag{7}$$

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Remark. The same method may be applied to the following system:

$$\begin{array}{l} y_{tt} - \Delta y - a\Delta y_t = 0 \text{ in } \Omega \times (0,\infty) \\ y = 0 \text{ on } \Sigma = \Gamma \times (0,\infty), \quad y(0) = y^0, \quad y_t(0) = y^1 \text{ in } \Omega. \end{array}$$

But now, the natural the energy space is $\hat{\mathcal{H}} = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

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4. The case of a nonlinear damping is open.
5. Extending the polynomial and exponential stability results to the optimal geometric condition of Bardos-Lebeau-Rauch is an open problem.
6. The analogous problem for the plate equation $y_{tt} + \Delta^2 y + \Delta(a\Delta y_t) = 0$ in $\Omega \times (0, \infty)$ with clamped BCs is open in the multidimensional setting. No smoothness on the damping coefficient is needed in the one-dimensional setting, (Liu-Liu, 1998).

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!