MAC 2311 (Calculus I)
TEST 1, Friday October 02, 2009

Name:

PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good Luck.

1. [40] Evaluate the following limits (Show all your work)

a) $\lim_{x\to 1} \frac{x^2-3x}{x^3-2x+6} = \frac{1-3}{1-2+6} = \frac{-2}{5}$

b) $\lim_{x\to+\infty} \cos\left(\frac{-\pi x^4+3x+7}{8-5x^2+2x^4}\right) = \cos\left(\lim_{x\to+\infty} \frac{-\pi x^4+3x+7}{8-5x^2+2x^4}\right) = \cos\left(\lim_{x\to+\infty} \frac{-\pi x^4}{2x^4}\right) = \cos\left(-\frac{\pi}{2}\right)$
$= \cos\left(\frac{\pi}{2}\right) = 0$

c) $\lim_{x\to -2^-} \frac{x}{x+2} = \lim_{x\to -2^-} x \cdot \lim_{x\to -2^-} \frac{1}{x+2} = -2(-\infty) = +\infty$

d) $\lim_{x\to 2} \frac{\sqrt{3x-2}-2}{x-2} = \lim_{x\to 2} \frac{(\sqrt{3x-2}-2)(\sqrt{3x-2}+2)}{(x-2)(\sqrt{3x-2}+2)} = \lim_{x\to 2} \frac{3x-2-4}{(x-2)(\sqrt{3x-2}+2)}$
$= \lim_{x\to 2} \frac{3(x-2)}{(x-2)(\sqrt{3x-2}+2)} = \lim_{x\to 2} \frac{3}{\sqrt{3x-2}+2} = \frac{3}{\sqrt{\lim_{x\to 2}(3x-2)}+2}$
$= \frac{3}{\sqrt{4}+2} = \frac{3}{4}$

e) $\lim_{x\to -3} \sqrt{\frac{3x^2-5x+3}{-5x-3}} = \sqrt{\lim_{x\to -3} \frac{3x^2-5x+3}{-5x-3}} = \sqrt{\frac{3(9)-5(-3)+3}{-5(-3)-3}} = \sqrt{\frac{45}{12}} = \sqrt{\frac{15}{4}}$
$= \frac{\sqrt{15}}{2}$

f) $\lim_{x\to \frac{\pi}{4}^-} \frac{\sin(2x)}{|x|} = \frac{\lim_{x\to \frac{\pi}{4}^-} \sin(2x)}{\lim_{x\to \frac{\pi}{4}^-} |x|} = \frac{\sin(\pi/2)}{\pi/4} = \frac{4(1)}{\pi} = \frac{4}{\pi}$

g) $\lim_{x\to 3} \frac{x^2-9}{x^2-5x+6} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x-2)} = \lim_{x\to 3} \frac{x+3}{x-2} = \frac{3+3}{3-2} = 6$

h) $\lim_{x\to -\infty} \frac{\sqrt{9x^2-5x+6}}{-5x+7} = \lim_{x\to -\infty} \frac{\sqrt{x^2(9-\frac{5}{x}+\frac{6}{x^2})}}{-x(5-\frac{7}{x})} = \lim_{x\to -\infty} \frac{\sqrt{x^2}\cdot\sqrt{9-\frac{5}{x}+\frac{6}{x^2}}}{-x(5-\frac{7}{x})}$
$= \lim_{x\to -\infty} \frac{|x|\sqrt{9-\frac{5}{x}+\frac{6}{x^2}}}{|x|(5-\frac{7}{x})} = \frac{\sqrt{9-\lim_{x\to-\infty}\frac{5}{x}+\lim_{x\to-\infty}\frac{6}{x^2}}}{5-7\lim_{x\to-\infty}\frac{1}{x}}$
$= \frac{\sqrt{9}}{5}$
$= \frac{3}{5}$

2. [7] If $f(x) = \begin{cases} 2x^2 - 3, & x > 1 \\ x^3 - 2, & x \leq 1. \end{cases}$

Is f continuous at $x = 1$? You must carefully explain your answer to get any credits.

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x^2 - 3) = 2(1) - 3 = -1$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^3 - 2) = (1^3) - 2 = 1 - 2 = -1$. So $\lim_{x \to 1} f(x) = -1 = f(1)$,

as $f(1) = 1^3 - 2 = -1$. So f is continuous at $x = 1$.

3. [10] Write down the rigorous definition of $\lim_{x \to a} f(x) = L$, and use it to prove that $\lim_{x \to -3} (5x + 6) = -9$.

$\lim_{x \to a} f(x) = L$ iff for every $\varepsilon > 0$, there exists $\delta > 0$: for every x with $0 < |x - a| < \delta$, one has $|f(x) - L| < \varepsilon$.

Let $\varepsilon > 0$. Find $\delta > 0$: for every x with $0 < |x + 3| < \delta$, one has $|5x + 6 - (-9)| < \varepsilon$. For $|5x + 6 - (-9)| < \varepsilon$, it is enough that $|5x + 15| < \varepsilon$, or $5|x + 3| < \varepsilon$, or $|x + 3| < \varepsilon/5$. It suffices to choose $\delta = \varepsilon/5$.

4. [6] a) Use the implicit differentiation technique to find $\frac{dy}{dx}$ if $x^2y^3 - 4x + 12y = 8$.
b) Find the equation of the tangent line to the curve $x^2y^3 - 4x + 12y = 8$ at the point (2,1).

a) $\frac{d}{dx}(x^2y^3 - 4x + 12y) = \frac{d}{dx}(8) = 0$

$2xy^3 + 3x^2y^2\frac{dy}{dx} - 4 + 12\frac{dy}{dx} = 0$

Hence $\frac{dy}{dx} = \frac{4 - 2xy^3}{3x^2y^2 + 12}$

b) slope of tangent line $= \frac{dy}{dx}\Big|_{x=2, y=1} = \frac{4 - 2(2)(1)}{3(1)(2^2) + 12} = \frac{4-4}{24} = 0$

Eqn of tangent line: $y = 1$; tangent line is horizontal.

5. [37] Find the derivative of each of the following functions (Show all your work, and simplify your answers as much as possible; you will not get any credit by guessing the correct answer(s).)

a) $f(x) = 2x^3 - \frac{5}{\sqrt[3]{x}} + \frac{7}{x^2}$ $= 2x^3 - 5x^{-\frac{1}{3}} + 7x^{-2}$

$$f'(x) = 6x^2 + \frac{5}{3}x^{-\frac{4}{3}} - 14x^{-3}$$

b) $g(x) = \frac{3x-4}{x^2+x+1}$

$$g'(x) = \frac{3(x^2+x+1) - (2x+1)(3x-4)}{(x^2+x+1)^2}$$
$$= \frac{3x^2+3x+3 - (6x^2-8x+3x-4)}{(x^2+x+1)^2}$$
$$= \frac{-3x^2+8x+7}{(x^2+x+1)^2}$$

c) $h(x) = x^2\ln(x)$

$$h'(x) = 2x\ln x + x^2\left(\frac{1}{x}\right)$$
$$= 2x\ln x + x$$
$$= x(2\ln x + 1)$$

d) $k(x) = \sin^2(\sec^3 x) + \cos^2(\sec^3 x)$

$= 1$ for every x in D_{sec}

So, $k'(x) = 0$

e) $l(x) = \tan(x^3 - \csc x)$

$$l'(x) = (3x^2 - (-\csc x\cot x))\sec^2(x^3 - \csc x)$$
$$= (3x^2 + \csc x\cot x)\sec^2(x^3 - \csc x)$$

f) $m(x) = \sin(\cos x)$

$$m'(x) = -\sin x\cos(\cos x)$$

g) Use the logarithmic differentiation technique to find the derivative of $p(x) = \frac{x^2\sqrt[5]{x^2-x+2}}{x^3+x-1}$.

Set $y = p(x)$. Then

$$\ln y = \ln p(x) = \ln(x^2(x^2-x+2)^{\frac{1}{5}}) - \ln(x^3+x-1)$$
$$= \ln(x^2) + \ln[(x^2-x+2)^{\frac{1}{5}}] - \ln(x^3+x-1)$$
$$= 2\ln x + \frac{1}{5}\ln(x^2-x+2) - \ln(x^3+x-1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}\left(2\ln x + \frac{1}{5}\ln(x^2-x+2) - \ln(x^3+x-1)\right)$$
$$= \frac{2}{x} + \frac{2x-1}{5(x^2-x+2)} - \frac{(3x^2+1)}{x^3+x-1}$$

$$p'(x) = p(x)\left(\frac{2}{x} + \frac{2x-1}{5(x^2-x+2)} - \frac{(3x^2+1)}{x^3+x-1}\right)$$

Bonus. [6] $\lim_{x\to-\infty}(\sqrt{4x^2-5x}+2x) =$ $\lim_{x\to-\infty}\frac{(\sqrt{4x^2-5x}+2x)(\sqrt{4x^2-5x}-2x)}{\sqrt{4x^2-5x}-2x}$

$$= \lim_{x\to-\infty}\frac{4x^2-5x-4x^2}{\sqrt{x^2}\sqrt{4-\frac{5}{x}}-2x}$$

$$= \lim_{x\to-\infty}\frac{-5x}{-x(\sqrt{4-\frac{5}{x}}+2)} = \frac{5}{\sqrt{4-\lim_{x\to-\infty}\frac{5}{x}}+2} = \frac{5}{\sqrt{4}+2} = \frac{5}{4},$$