MAC 2311 (Calculus I) — Key
Test 1, Friday October 18, 2013

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University administration. Good Luck!

1. [30] Evaluate the following limits (Show all your work. You cannot use de l'Hopital's rule for any of these limits, otherwise you will not get any credit(s). Besides, you will not get any credit(s) by guessing the correct answer(s).)

a) $\lim\limits_{x\to 2}\sqrt[3]{\frac{x^3-5x+1}{x^2+3x-1}} = \sqrt[3]{\frac{2^3-5(2)+1}{2^2+3(2)-1}} = \sqrt[3]{\frac{-1}{9}}$

b) $\lim\limits_{x\to+\infty}\frac{2x^3-17x+2013^{1018}}{2013^{1018}+x^2-5x^3} = \lim\limits_{x\to+\infty}\frac{2\cancel{x^3}}{-5x^3} = -\frac{2}{5}$

c) $\lim\limits_{x\to\frac{\pi}{6}}\frac{\sin x}{x} = \frac{\sin\frac{\pi}{6}}{\pi/6} = \frac{1/2}{\pi/6} = \frac{6/2}{\pi} = \frac{3}{\pi}$

d) $\lim\limits_{x\to 6}\frac{\sqrt{x-2}-2}{x^2-36} = \lim\limits_{x\to 6}\frac{(\sqrt{x-2}-2)(\sqrt{x-2}+2)}{(x-6)(x+6)(\sqrt{x-2}+2)}$

$= \lim\limits_{x\to 6}\frac{\cancel{x-2-4}}{\cancel{(x-6)}(x+6)\sqrt{x-2}+2}$

$= \frac{1}{(6+6)\sqrt{6-2}+2} = \frac{1}{12(4)} = \frac{1}{48}$

e) $\lim\limits_{x\to -2^-}\frac{2x-3}{x^2-4} = \lim\limits_{x\to -2^-}\frac{2x-3}{(x-2)(x+2)}$

$= \lim\limits_{x\to -2^-}\frac{2x-3}{x-2}\cdot\lim\limits_{x\to -2^-}\frac{1}{x+2}$

$= \frac{2(-2)-3}{-2-2}\cdot(-\infty)$

$= \frac{-7}{-4}(-\infty) = \frac{7}{4}(-\infty) = -\infty$

f) $\lim\limits_{x\to 0}\frac{2x^2}{1-\cos x} = \lim\limits_{x\to 0}\frac{2x^2(1+\cos x)}{(1-\cos x)(1+\cos x)}$

$= \lim\limits_{x\to 0}\frac{2x^2(1+\cos x)}{1-\cos^2 x}$

$= 2\lim\limits_{x\to 0}\frac{x^2}{\sin^2 x}\cdot\lim\limits_{x\to 0}(1+\cos x)$

$= 2\cdot(1)\cdot(1+1) = 4$

2. [10] Let $y = 2x^2-3$. a) find the average rate of change of y with respect to x on the interval $[-1,2]$. b) Find the instantaneous rate of change of y with respect to x at $x_0=-1$.

a) $y_{average} = \frac{y(2)-y(-1)}{2-(-1)} = \frac{(2(4)-3)-(2(-1)^2-3)}{3} = \frac{8-3-(-1)}{3} = \frac{6}{3} = 2$

b) $y_{inst.} = \lim\limits_{h\to 0}\frac{y(-1+h)-y(-1)}{h} = \lim\limits_{h\to 0}\frac{2(-1+h)^2-3-(-1)}{h} = \lim\limits_{h\to 0}\frac{2(\cancel{1}-2h+h^2)-\cancel{2}}{h}$

$= \lim\limits_{h\to 0}\frac{-4h+2h^2}{h} = \lim\limits_{h\to 0}(-4+2h) = -4$

3. [10] a) Write down the two definitions for $f'(x_0)$. b) Use any of those definitions to find $f'(1)$ if $f(x)=\sqrt{x}$. c) Use part b) to find the equation of the tangent line to the curve $y=\sqrt{x}$ at $x=1$.

a) i) $f'(x_0) = \lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$, ii) $f'(x_0) = \lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$

b) using i), $f'(1) = \lim\limits_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim\limits_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}+1)(x-1)} = \lim\limits_{x\to 1}\frac{\cancel{x-1}}{(\sqrt{x}+1)\cancel{(x-1)}}$

$= \lim\limits_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{1+1} = \frac{1}{2}$

c) $y = f(1)+f'(1)(x-1) = 1+\frac{1}{2}(x-1)$

4. [10] a) State the intermediate-value theorem. b) Use that theorem to show that the equation $x^{18}-3x+1=0$ has a unique solution in the open interval $(0,1)$.

a) See notes or text.

b) Set $f(x) = x^{18}-3x+1$. f is a polynomial function, so f is continuous on $[0,1]$. Now $f(0)\cdot f(1) = 1(1-3+1) = -1 < 0$; so f has at least one root in $(0,1)$ by the intermediate value theorem, that means the given equation has at least one solution in $(0,1)$.

5. [32] Find the derivative of each of the following functions (Show all your work, and simplify your answers as much as possible; you will not get any credit(s) by guessing the correct answer(s).)

a) $f(x) = \frac{2x}{x^2-3x+1}$

$$f'(x) = \frac{2(x^2-3x+1)-(2x-3)2x}{(x^2-3x+1)^2}$$
$$= \frac{2x^2-\cancel{6x}+2-4x^2+\cancel{6x}}{}$$
$$= \frac{-2x^2+2}{(x^2-3x+1)^2}$$

b) $g(x) = x^3\ln(1+x^2)$

$$g'(x) = 3x^2\ln(1+x^2) + x^3 \cdot \frac{2x}{(1+x^2)}$$
$$= 3x^2\ln(1+x^2) + \frac{2x^4}{1+x^2}$$

c) $h(x) = \tan^{-1}(x^4)$

$$h'(x) = 4x^3 \cdot \frac{1}{1+(x^4)^2} = \frac{4x^3}{1+x^8}$$

d) $k(x) = 3^{\cos x}$, $k'(x) = -\sin x \cdot 3^{\cos x} \cdot \ln 3$
$$= -(\sin x) \cdot 3^{\cos x} \cdot \ln 3$$

e) Use the logarithmic differentiation technique to find $\frac{dy}{dx}$ if $y = (x+\tan x)^{\sin x}$.

$$\ln y = \ln[(x+\tan x)^{\sin x}] = \sin x \ln(x+\tan x)$$
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x \cdot \ln(x+\tan x)) = \cos x \cdot \ln(x+\tan x) + \sin x \cdot \frac{(1+\sec^2 x)}{x+\tan x}$$
$$\frac{1}{y}\frac{dy}{dx} = (\cos x)\ln(x+\tan x) + \sin x \cdot \frac{(1+\sec^2 x)}{x+\tan x}$$
Hence $$\frac{dy}{dx} = \left[(\cos x)\ln(x+\tan x) + \frac{\sin x(1+\sec^2 x)}{x+\tan x}\right](x+\tan x)^{\sin x}.$$

f) Use the implicit differentiation technique to find $\frac{dy}{dx}$ if $xy + x^2\cos(y) = 1$.

$$\frac{d}{dx}(xy + x^2\cos y) = \frac{d}{dx}(1) = 0$$
$$y + x\frac{dy}{dx} + 2x\cos y + x^2(-\sin y)\frac{dy}{dx} = 0$$
$$(x - x^2\sin y)\frac{dy}{dx} = -(y+2x\cos y)$$
$$\frac{dy}{dx} = \frac{-(y+2x\cos y)}{x(1-x\sin y)}$$

6. [14] Decide whether the statement is true or false. No explanation needed.

a) If $f(x) = \frac{\sin x}{g(x)}$, then $f'(x) = \frac{\cos x}{g'(x)}$. False, $f'(x) = ((\cos x)g(x) - g'(x)\sin x)/(g(x))^2$

b) If $|f|$ is continuous at 3, then f is continuous at 3. False; set $f(x) = \begin{cases} -1, & x \le 3 \\ 1, & x > 3 \end{cases}$, then $|f(x)| = 1$ for all x

c) If f is continuous at x_0, then $\lim_{x\to x_0} f(x) = f(x_0)$. True

d) If f is differentiable at -7, then f is continuous at -7. True

e) If $\lim_{x\to 4^+} f(x) = 18$ and $\lim_{x\to 4^-} f(x) = 18$, then f is continuous at 4. False

f) If $\lim_{x\to 5^+} f(x) = -\infty$, then $f(5)$ is undefined. False; set $f(x) = \begin{cases} \frac{1}{5-x}, & x \ne 5 \\ -2, & x = 5 \end{cases}$

g) $\lim_{x\to 0^-}(\frac{1}{x^2}+\frac{1}{x}) = +\infty + (-\infty) = 0$. False, $\lim_{x\to +\infty}\frac{1}{x^2}+\frac{1}{x} = \lim_{x\to 0^-}\frac{1}{x^2}(1+x) = \lim_{x\to 0^-}\frac{1}{x^2}\cdot\lim_{x\to 0^-}(1+x)$
$= +\infty(1)$
$= +\infty$