MAC 2311 (Calculus I)
TEST 1, Friday October 6, 2017 — Answers

Name:

PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may show your work on the back of page. Total=86 points. Always do your best.

1. [24] Evaluate the following limits (Show all your work. You will not get any credit(s) by guessing the correct answer(s). You cannot use de l'Hopital's rule for any of the limits, otherwise you'll get a zero. But you may use your knowledge of the derivative. If a limit is infinite, clearly state whether it is $+\infty$ or $-\infty$.)

a) $\lim_{x\to-2}\frac{5-4x}{x^2+3x-5} = \frac{5-4(-2)}{(-2)^2+3(-2)-5} = \frac{5+8}{4-6-5} = \frac{13}{-7} = -\frac{13}{7}$

b) $\lim_{x\to-\infty}\frac{5x^2-7x+3^{106}}{4^{106}-3x^2} = \lim_{x\to-\infty}\frac{5\cancel{x^2}}{-3\cancel{x^2}} = -\frac{5}{3}$

c) $\lim_{x\to1^+}\frac{x-2}{x^2-1} = \lim_{x\to1^+}\frac{x-2}{(x+1)(x-1)} = \lim_{x\to1^+}\frac{x-2}{x+1}\cdot\lim_{x\to1^+}\frac{1}{x-1} = \frac{1-2}{1+1}\cdot(+\infty) = -\frac{1}{2}(+\infty) = -\infty$

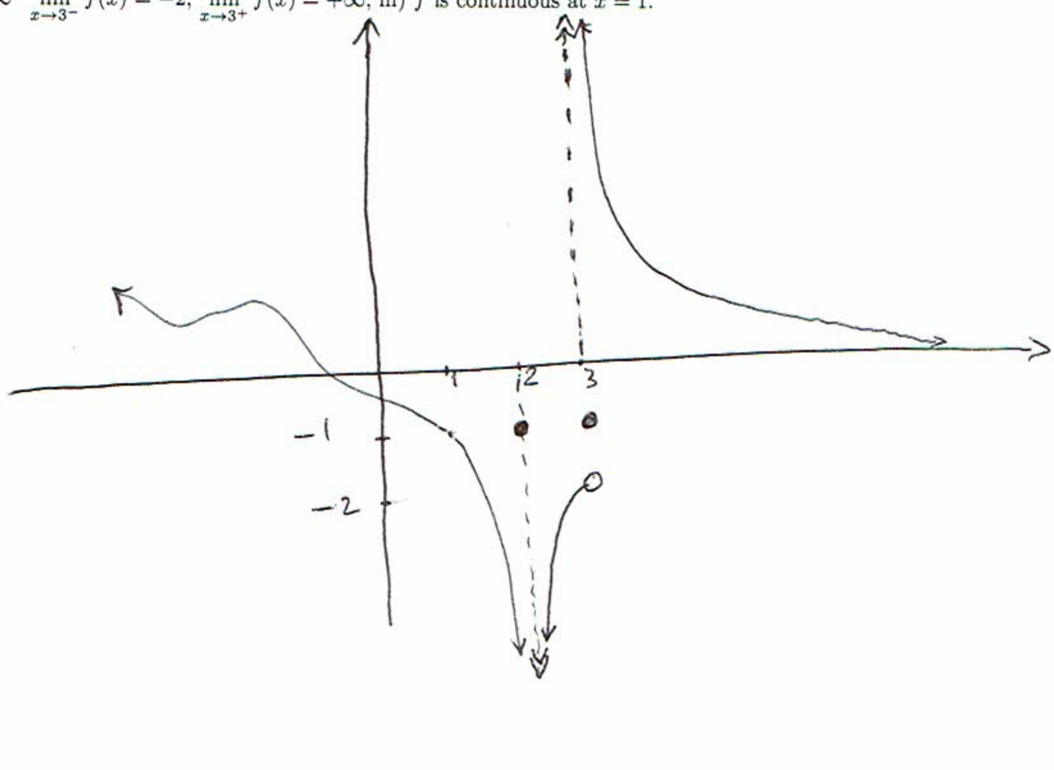
d) $\lim_{x\to-1}\frac{x^4+2x+1}{x^3+1} = \lim_{x\to-1}\frac{\cancel{(x+1)}(x^3-x^2+x+1)}{\cancel{(x+1)}(x^2-x+1)} = \lim_{x\to-1}\frac{x^3-x^2+x+1}{x^2-x+1} = \frac{-1-1-1+1}{1+1+1} = -\frac{2}{3}$

e) $\lim_{x\to0}\frac{\sin(8x)}{\sin(5x)} = \lim_{x\to0}\frac{\frac{\sin(8x)}{8x}\cdot 8\cancel{x}}{\frac{\sin(5x)}{5x}\cdot5\cancel{x}} = \frac{8}{5}\frac{\lim_{x\to0}(\sin(8x)/8x)}{\lim_{x\to0}(\sin(5x)/5x)} = \frac{8}{5}\cdot\frac{1}{1} = \frac{8}{5}$

f) $\lim_{x\to+\infty}x\sin(1/6x) = \lim_{x\to+\infty}\frac{\sin(1/6x)}{\frac{1}{6x}\cdot6}$, set $u=\frac{1}{6x}$

$= \lim_{u\to0^+}\frac{\sin u}{u(6)} = \frac{1}{6}\lim_{u\to0^+}\frac{\sin u}{u} = \frac{1}{6}$

2. [6, **Bonus**] Sketch the graph of a function f that satisfies all the properties: i) $f(1)=-1=f(2)=f(3)$, ii) $\lim_{x\to2}f(x)=-\infty$ $\lim_{x\to3^-}f(x)=-2$, $\lim_{x\to3^+}f(x)=+\infty$, iii) f is continuous at $x=1$.



3. [10] a) Write down the rigorous definition of $\lim_{x \to -2} f(x) = -8$. b) Use the rigorous definition of limit to prove that $\lim_{x \to -2}(3x-2) = -8$.

a) For every $\varepsilon > 0$, there exists $\delta > 0$: $|f(x)+8| < \varepsilon$ whenever $0 < |x+2| < \delta$.

b) Let $\varepsilon > 0$. Find $\delta > 0$: $|3x-2+8| < \varepsilon$ whenever $0 < |x+2| < \delta$

For $|3x-2+8| < \varepsilon$, it's enough that $|3x+6| < \varepsilon$

or $|3(x+2)| < \varepsilon$

or $3|x+2| < \varepsilon$

or $|x+2| < \varepsilon/3$

We may choose $\delta = \varepsilon/3$.

4. [10] a) State the intermediate value theorem (IVT). b) Use it to show that the equation $(x-1)^{106} - 6(x-1)^{11} + 2 = 0$ has a solution in the open interval $(1, 2)$.

a) Let f be continuous on the closed interval $[a,b]$.

Assume that $f(a) \cdot f(b) < 0$.

There exists x_0 in the open interval (a,b) with $f(x_0) = 0$.

b) Set $f(x) = (x-1)^{106} - 6(x-1)^{11} + 2$

- f is a polynomial, so f is continuous everywhere. f is continuous on the closed interval $[1,2]$, as f is continuous everywhere.
- $f(1) \cdot f(2) = 2(1-6+2) = 2(-3) = -6 < 0$

So there exists x_0 in $(1,2)$ with $(x_0-1)^{106} - 6(x_0-1)^{11} + 2 = 0$, by the IVT.

5. [6] Write down the correct statement.

a) Constant rule: Let k be a constant. Then $\frac{d}{dx}(k) = 0$

b) Power rule: $\frac{d}{dx}(x^r) = r x^{r-1}$, $r =$ constant

c) Product rule: $(fg)'(x) = f'(x)g(x) + g'(x)f(x)$

d) Quotient rule: $(f/g)'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

e) Chain rule: $(f(g))'(x) = g'(x) f'(g(x))$

f) Write down the definition: $f'(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ or $f'(x_0) = \lim\limits_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$

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6. [10] Decide whether the statement is true or false. No explanation needed.

a) If $k(x) = \frac{g(x)}{x^3}$, then $k'(x) = \frac{3x^2 g(x) - g'(x)x^3}{x^6}$. F, $k'(x) = \dfrac{g'(x)x^3 - 3x^2 g(x)}{x^6}$

b) If $f(-3) = 5$, then $\lim\limits_{x \to -3} f(x) = 5$. F, f may be defined without the limit being necessarily the same as the value of f

c) If $p(x) = q(x^2)$, then $p'(x) = q'(2x)$. F, $p'(x) = 2x q'(x^2)$

d) If f is differentiable at -8, then f is continuous at -8. T, Theorem 2.2.3.

e) If f is differentiable at $x = 2$, then $f'(2)$ is the instantaneous rate of change of f with respect to x at $x = 2$. T, See 2.1 & 2.2

f) If $g(x) = m(x)x^5$, then $g'(x) = m'(x)x^5 + 5x^4 m(x)$. T, by the Product Rule

g) If $\lim\limits_{x \to 1^+} f(x) = 26$ and $\lim\limits_{x \to 1^-} f(x) = 26$, then f is continuous at $x = 1$. False, f might not be defined at $x=1$.

h) If $g(x) = 6^{10}$, then $g'(x) = 10(6^9)$. F, $g'(x) = 0$, by the Constant Rule

i) If $k(x) = \csc^2(\sin^3(\cos x)) - \cot^2(\sin^3(\cos(x)))$, then $k'(x) = 0$. True, $k(x) = 1$ for all x, so $k'(x) = 0$

j) If $\lim\limits_{h \to 0} \dfrac{f(-7+h) - f(-7)}{h} = 4$, then $\lim\limits_{x \to -7} \dfrac{f(x) - f(-7)}{x+7} = 4$. True by definition in Notes

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7. [20] Find the derivative of each of the following functions (Show all your work, and simplify your answers as much as possible; you will not get any credit(s) by guessing the correct answer(s).)

a) $f(x) = \cos(\sin x)$

$$f'(x) = \cos x\,(-\sin(\sin x))$$
$$= -\cos x \sin(\sin x)$$

b) $g(x) = x^3 \sin(2x)$

$$g'(x) = 3x^2 \sin(2x) + 2x^3 \cos(2x)$$

c) $h(x) = \frac{x^2+3}{x^2+2x-5}$

$$h'(x) = \frac{2x(x^2+2x-5) - (2x+2)(x^2+3)}{(x^2+2x-5)^2}$$
$$= \frac{\cancel{2x^3} + 4x^2 - 10x - (\cancel{2x^3} + 6x + 2x^2 + 6)}{(x^2+2x-5)^2}$$
$$= \frac{4x^2 - 2x^2 - 10x - 6x - 6}{(x^2+2x-5)^2}$$
$$= \frac{2x^2 - 16x - 6}{(x^2+2x-5)^2}$$

d) $k(x) = -\frac{3}{x^4} - \frac{8}{x^{\frac{1}{4}}} + \frac{2}{\sqrt[5]{x^3}} - 5^{11}$ $= -3x^{-4} - 8x^{-\frac{1}{4}} + 2x^{-\frac{3}{5}} - 5^{11}$

$$k'(x) = 12x^{-5} + \frac{8}{4}x^{-\frac{5}{4}} - \frac{6}{5}x^{-\frac{8}{5}} - 0$$
$$= \frac{12}{x^5} + \frac{2}{\sqrt[4]{x^5}} - \frac{6}{5\sqrt[5]{x^8}}$$