MAC 2312 (Calculus II) — Key
Test 1, Monday October 14, 2013

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; no credits will be awarded for unexplained answers. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good luck.

1. [6] Determine whether the improper integral converges or diverges. If it converges, state its limit, and if it diverges, state whether it diverges to $+\infty$ or to $-\infty$.

a) $\int_1^2 \frac{dx}{(x-1)^{\frac{7}{6}}}\,dx = \lim_{a\to 1^+} \int_a^2 (x-1)^{-7/6}dx = \lim_{a\to 1^+} -\frac{6}{(x-1)^{1/6}}\Big]_a^2 = -6 + 6\lim_{a\to 1^+}\frac{1}{(a-1)^{1/6}}$
$= -6 + 6(+\infty)$
$= +\infty$

So I diverges to $+\infty$.

2. [6] Solve the initial-value problem: $\begin{cases} \frac{dy}{dx} = \frac{2x}{x^2+1} \\ y(0) = -2. \end{cases}$

$y = y(0) + \int_0^x \frac{2t}{t^2+1}dt = -2 + \int_{0+1}^{x^2+1}\frac{du}{u} = -2 + \ln u\Big]_1^{x^2+1}$
$= -2 + \ln(x^2+1) - \underbrace{\ln 1}_{0}$

$u = t^2+1$
$du = 2t\,dt$

$y(x) = -2 + \ln(x^2+1).$

2. [10] a) Use the difference $a_{n+1} - a_n$ to show that the sequence $(a_n)_n$ given by: $a_n = \frac{3n+5}{2n+3}$, $n = 0, 1, \ldots$, is strictly decreasing. b) Show that the sequence $(a_n)_n$ is bounded from below, and derive that $(a_n)_n$ converges. c) Find its limit.

a) $a_{n+1} - a_n = \frac{3(n+1)+5}{2(n+1)+3} - \frac{(3n+5)}{2n+3} = \frac{(3n+8)(2n+3) - (3n+5)(2n+5)}{(2n+5)(2n+3)}$
$= \frac{\cancel{6n^2} + \cancel{9n} + \cancel{16n} + 24 - (\cancel{6n^2} + \cancel{15n} + \cancel{10n} + 25)}{(2n+5)(2n+3)}$
$= \frac{-1}{(2n+5)(2n+3)} < 0$, for all $n \geq 0$. So (a_n) is strictly decreasing.

b) Since $3n+5 > 0$ and $2n+3 > 0$ for all $n \geq 0$, we have $a_n > 0$ for all $n \geq 0$.
Now (a_n) is strictly decreasing, and bounded from below,
So (a_n) converges.

c) $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty}\frac{3n+5}{2n+3} = \lim_{n\to+\infty}\frac{n(3+\frac{5}{n})}{n(2+\frac{3}{n})} = \lim_{n\to+\infty}\frac{3+\frac{5}{n}}{2+\frac{3}{n}} = \frac{3}{2}.$

3. [10] Decide whether each statement is true or false. No explanation is needed.

a) If (a_n) is a bounded sequence, then (a_n) converges. False, $a_n = (-1)^n$, $n = 1, 2, \dots$

b) If f is integrable on $[a, b]$, then f is continuous on $[a, b]$. False, example in notes

c) If f is integrable on $[1, 2]$, then $\int_1^2 f(x)\,dx + \int_2^1 f(x)\,dx = 0$. True, Property of definite integral

d) If (a_n) is an increasing sequence, then (a_n) converges. False, $a_n = n$, $n = 1, 2, \dots$

e) $\int_{-1}^{1} \frac{dx}{x^4} = -\frac{2}{3}$. False, integral is improper and FTC1 does not apply

4. [8] Approximate the integral $\int_0^{\sqrt{\pi}} \sin(x^2)\,dx$ using: a) the trapezoidal rule with $n = 2$. b) Simpson's rule with $n = 2$.

$x_0 = 0, \quad x_1 = \frac{\sqrt{\pi}}{2}, \quad x_2 = \sqrt{\pi}$

a) $$\int_0^{\sqrt{\pi}} \sin(x^2)dx \approx T_2 = \frac{\sqrt{\pi}}{4}(\underbrace{\sin(0)}_{=0} + 2\sin(\tfrac{\pi}{4}) + \underbrace{\sin(\pi)}_{=0})$$
$$= \frac{\sqrt{\pi}}{4} \cdot 2\frac{\sqrt{2}}{2} = \frac{\sqrt{\pi}\sqrt{2}}{4} = \frac{\sqrt{2\pi}}{4}$$

b) $$\int_0^{\sqrt{\pi}} \sin(x^2)dx \approx S_2 = \frac{\sqrt{\pi}}{6}(\sin(0) + 4\sin(\tfrac{\pi}{4}) + \sin(\pi))$$
$$= \frac{\sqrt{\pi}}{6} \cdot 4\frac{\sqrt{2}}{2} = \frac{\sqrt{2\pi}}{3}$$

5. [25] Evaluate each definite integral. (Show all your work. No credits if no work is shown.)

a) $\int_0^{\frac{\pi}{6}} \sec^2 x\,dx = \tan x\Big]_0^{\pi/6} = \tan\frac{\pi}{6} - \underbrace{\tan 0}_{=0} = \frac{1}{\sqrt{3}}$

b) $\int_{-1}^{1} |1 - e^x|\,dx = \int_{-1}^{0} |1-e^x|dx + \int_0^1 |1-e^x|dx = \int_{-1}^{0} 1-e^x dx - \int_0^1 1-e^x dx$
$= x-e^x\Big]_{-1}^{0} - (x-e^x)\Big]_0^1$
$= 0-1-(-1-e^{-1}) - [(1-e)-(0-1)]$
$= -1+1+e^{-1}-1+e-1$
$= e+e^{-1}-2$

$1-e^x=0 \to x=0$

c) $\int_0^4 \frac{dx}{16+x^2} = \int_0^1 \frac{4du}{16(u^2+1)} = \frac{1}{4}\tan^{-1}u\Big]_0^1$
$= \frac{1}{4}(\tan^{-1}1 - \underbrace{\tan^{-1}0}_{=0})$
$= \frac{\pi}{16}$

$x = 4u$
$dx = 4du$

d) $\int_1^3 x^2 \ln x\,dx = \frac{x^3}{3}\ln x\Big]_1^3 - \int_1^3 \frac{x^3}{3}\frac{dx}{x} = \frac{x^3}{3}\ln x\Big]_1^3 - \frac{x^3}{9}\Big]_1^3$
$= 9\ln 3 - \frac{(27-1)}{9}$, as $\ln 1 = 0$
$= 9\ln 3 - \frac{26}{9}$

$u = \ln x \quad dv = x^2dx$
$du = \frac{dx}{x} \quad v = \frac{x^3}{3}$

e) $\int_0^{\frac{\pi}{4}} \sqrt{\sec x}\tan x \sec x\,dx = \int_1^{\sqrt{2}} \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big]_1^{\sqrt{2}}$
$= \frac{2}{3}(2\sqrt{2}-1)$

$u = \sec x$
$du = \sec x \tan x dx$

6. [25] Evaluate each indefinite integral. (Show all your work. No credits if no work is shown.)

i) $\int x\sqrt{x-2}\,dx =$ $\int (u+2)\sqrt{u}\,du = \int u^{3/2} + 2u^{1/2}\,du = \frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} + C$

$u = x-2,\ du = dx$

$x = u+2$

$= \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$

j) $\int \frac{dx}{x^4+2x^2+1} =$ $\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,d\theta = \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta\,d\theta$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$= \frac{1}{2}\int \cos(2\theta)+1\,d\theta = \frac{1}{2}\left(\frac{\sin(2\theta)}{2} + \theta\right) + C$

$= \frac{1}{2}(\sin\theta\cos\theta + \theta) + C$

$= \frac{1}{2}\left(\frac{x}{x^2+1} + \tan^{-1}x\right) + C$

k) $\int \frac{dx}{(x+3)(x^2+1)} =$ $\int \frac{a}{x+3} + \frac{bx+d}{x^2+1}\,dx$

$= a\ln|x+3| + \frac{b}{2}\ln(x^2+1) + d\tan^{-1}x + C$

$= \frac{1}{10}\ln|x+3| - \frac{1}{20}\ln(x^2+1) + \frac{3}{10}\tan^{-1}x + C$

$a(x^2+1) + (bx+d)(x+3) = 1$

For $x = -3$: $10a = 1 \rightarrow a = 1/10$

$x = i\ (i^2 = -1)$: $(bi+d)(i+3) = 1 \rightarrow -b + 3bi + di + 3d = 1 \rightarrow$

$3b + d = 0 \quad (1)$

$-b + 3d = 1 \quad (2)$

$(1) + 3\cdot(2)$: $10d = 3 \rightarrow d = 3/10$

From (1), $b = -d/3 = -\frac{1}{10}$

$b = -\frac{1}{10}$

l) $\int \frac{3x^2+2}{x^2+1}\,dx =$ $\int \frac{3(x^2+1) - 1}{x^2+1}\,dx$

$= \int 3 - \frac{1}{x^2+1}\,dx = 3x - \tan^{-1}x + C$

m) $\int \sin(2x)\sin(3x)\,dx =$ $\int \frac{\cos(3x-2x) - \cos(3x+2x)}{2}\,dx$

$= \frac{1}{2}\int \cos x - \cos(5x)\,dx = \frac{1}{2}\left(\sin x - \frac{\sin(5x)}{5}\right) + C$

7. [10] a) Find the derivative $F'(x)$ if $F(x) = \int_{\sin x}^{x^3} \cos^{1014}(t + e^{2t})\,dt$.

a) $F'(x) = \frac{d}{dx}(x^3)\cos^{1014}(x^3 + e^{2x^3}) - \frac{d}{dx}(\sin x)\cdot\cos^{1014}(\sin x + e^{2\sin x})$

$= 3x^2\cos^{1014}(x^3 + e^{2x^3}) - \cos x \cdot \cos^{1014}(\sin x + e^{2\sin x}$

b) Use the definition of the definite integral to write $\int_0^{\frac{\pi}{3}} x^2\cos(x - \sec x)\,dx$ as limit of a Riemann sum. Do not evaluate the integral.

$\int_0^{\pi/3} x^2\cos(x - \sec x)\,dx = \lim_{\max\Delta x_k \to 0} \sum_{k=1}^{n} x_k^*\cos(x_k^* - \sec(x_k^*))\Delta x_k$; $a = 0$, $b = \frac{\pi}{3}$

8. [5, **Bonus**] State the fundamental theorem of calculus, part 1.

See notes or text.