MAC 2312 (Calculus II) — Answers
Test 1, Wednesday September 16, 2015

Name: **PID:**

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve any of the points assigned to any question. *Answers without the steps leading to them won't get any credits.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good luck.**

1. [20] Computation of integrals; exact value shall be provided for each integral.

a) $\int_1^2 \frac{1}{x}\ln x\,dx = \int_{\ln 1}^{\ln 2} u\,du = \frac{u^2}{2}\Big]_0^{\ln 2} = \frac{(\ln 2)^2}{2}$

$u = \ln x$
$du = \frac{dx}{x}$

b) $\int_1^3 |4-x^2|\,dx = \int_1^2 4-x^2\,dx + \int_2^3 -(4-x^2)\,dx = 4x - \frac{x^3}{3}\Big]_1^2 + \frac{x^3}{3} - 4x\Big]_2^3$
$= 8 - \frac{8}{3} - (4-\frac{1}{3}) + (9-12) - (\frac{8}{3}-8)$

$4-x^2=0 \rightarrow x = \pm 2$

c) $\int_0^{\frac{\pi}{3}} (\sin x - \frac{3}{\cos^2 x})\,dx = -\cos x - 3\tan x\Big]_0^{\pi/3} = -\cos\frac{\pi}{3} + \cos 0 - 3\tan\frac{\pi}{3} + 3\tan 0$
$= -\frac{1}{2} + 1 - 3\sqrt{3} = \frac{1}{2} - 3\sqrt{3}$

d) If $f(x) = \frac{x^3-x}{\sqrt{1-x^2}}$, find the average value of f on $[0,1]$.

$f_{ave} = \frac{1}{1-0}\int_0^1 \frac{x^3-x}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{-x(-x^2+1)}{\sqrt{1-x^2}}\,dx = \int_0^1 -x\sqrt{1-x^2}\,dx = \frac{1}{2}\int_1^0 \sqrt{u}\,du$
$= \frac{u^{3/2}}{3}\Big]_1^0 = 0 - \frac{1}{3} = -\frac{1}{3}$

$u = 1-x^2$
$du = -2x\,dx$

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2. [7] If $F(x) = \frac{d}{dx}\int_1^{e^{3x}} \ln(1+2^t)\,dt$, use part 2 of the fundamental theorem of Calculus to find its derivative $F'(x)$. What is $F(0)$?

$F(x) = [\frac{d}{dx}(e^{3x})]\ln(1+2^{e^{3x}}) = 3e^{3x}\ln(1+2^{e^{3x}})$

$F'(x) = 9e^{3x}\ln(1+2^{e^{3x}}) + \frac{(9e^{6x}2^{e^{3x}}\ln 2)}{1+2^{e^{3x}}}$

$F(0) = 3\ln(1+2) = 3\ln 3$

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3. [3] If $\int_{-5}^3 f(x)\,dx = -9$, then $\int_0^2 f(3-4x)\,dx = \int_3^{-5} f(u)(-\frac{du}{4}) = \frac{1}{4}\int_{-5}^3 f(u)du = -\frac{9}{4}$

$u = 3-4x$
$du = -4\,dx$

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4. [**Bonus** 10] a) Given that $a = 0$ and $b = 1$, use those values to express the following limit as an integral, but do not evaluate the integral:

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \cos^3(\frac{k\pi}{n}) = \int_0^1 \cos^3(\pi x)\,dx$$

b) Write the integral $\int_0^{\frac{\pi}{6}} \sin(e^x)\,dx$ as the limit of a Riemann sum, but do not evaluate the integral.

$$\int_0^{\pi/6} \sin(e^x)dx = \lim_{\max\Delta x_k \to 0} \sum_{k=1}^{n} \sin(e^{x_k^*})\Delta x_k$$

$$= \lim_{n\to\infty} \frac{\pi}{6n}\sum_{k=1}^{n} \sin(e^{(\frac{k\pi}{6n})}), \quad \Delta x = \frac{\pi}{6n}, \quad x_k^* = x_k = \frac{k\pi}{6n}$$

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5. [5] a) Use the sigma notation to express (do not evaluate): $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} = \sum_{k=1}^{6} \frac{(-1)^k}{2k-1}$

b) Evaluate $\sum_{k=1}^{5} k(5-2k) = 1(5-2)+2(5-4)+3(5-6)+4(5-8)+5(5-10)$

$$= +\cancel{3} + 2 - \cancel{3} - 12 - 25$$

$$= -35$$

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6. [8] a) State the first part of the Fundamental Theorem of Calculus (FTC1).

If f is continuous on $[a,b]$, and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

b) Can we use FTC1 to evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin^2 x}$? Explain your answer or no credit.

No, as $x \mapsto \frac{1}{\sin^2 x}$ is not continuous at the right of $x=0$, and it is unbounded.

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7. [7] A projectile is fired vertically upward with the initial velocity of 16ft/s from a tower 160ft high. The acceleration due to gravity is $g = 32\text{ft/s}^2$. a) How long will it take the projectile to reach its maximum point? b) What is the maximum height?

$v_0 = 16ft/s$, $\quad v(t) = -gt + v_0 = -32t + 16$

a) The projectile reaches its highest point when $v(t_1) = 0$ for some $t_1 > 0$. $v(t_1) = 0 \rightarrow 16 = 32t_1$; so $t_1 = \frac{1}{2}$ s

b) $s(t) = \int v(t)dt = -\frac{g}{2}t^2 + v_0 t + s_0$

$$= -16t^2 + 16t + 160$$

$$s(t_1) = -16(\tfrac{1}{4}) + 16(\tfrac{1}{2}) + 160$$

$$= -4 + 8 + 160$$

$$= 164\ ft = \text{maximum height}$$