MAC 2312 (Calculus II) — Answers
Test 1, Wednesday September 21, 2016

Name: **PID:**

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve any of the points assigned to any question. *Answers without the steps leading to them won't get any credits.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Always do your best.**

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1. [10] Use a Riemann sum involving subintervals of equal width with x_k^* as the right endpoint to find the area under the curve $y = 2x^2 + 3x + 1$ over theinterval $[0, 1]$.

$\left(\text{Remember that } \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \text{ and } \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.\right)$

$0 = x_0 < x_1 < \cdots < x_n = 1$

$\Delta x = x_k - x_{k-1} = \frac{1}{n}$

$x_k = \frac{k}{n}, \quad k = 0, 1, 2, \ldots, n$

$x_k^* = x_k = \frac{k}{n}$

If A denotes the area, then

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left[2\left(\frac{k}{n}\right)^2 + 3\left(\frac{k}{n}\right) + 1\right]\frac{1}{n}$$

$$= \lim_{n\to\infty} \left(\frac{2}{n^3}\sum_{k=1}^{n} k^2 + \frac{3}{n^2}\sum_{k=1}^{n} k + \frac{1}{n}\sum_{k=1}^{n} 1\right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n^2}\,\frac{n(n+1)}{2} + \frac{1}{n}(n)\right)$$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{\cancel{n^3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{\cancel{n^3}} + \frac{3}{2}\lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)}{n^2} + 1 = \frac{1}{3}(2) + \frac{3}{2} + 1, \text{ as } \lim_{n\to\infty}\frac{1}{n} = 0$$

$$= \frac{2}{3} + \frac{3}{2} + 1$$

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2. [10] a) Evaluate $\sum_{k=2}^{19}\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) = \sum_{k=2}^{19}\frac{1}{k^2} - \sum_{k=2}^{19}\frac{1}{(k+1)^2}$. Now $\sum_{k=2}^{19}\frac{1}{(k+1)^2} = \sum_{k=3}^{20}\frac{1}{k^2}$

$$= \sum_{k=2}^{19}\frac{1}{k^2} - \sum_{k=3}^{20}\frac{1}{k^2}$$

$$= \frac{1}{4} + \cancel{\sum_{k=3}^{19}\frac{1}{k^2}} - \cancel{\sum_{k=3}^{19}\frac{1}{k^2}} - \frac{1}{20^2} = \frac{1}{4} - \frac{1}{400} = \frac{99}{400}$$

b) Use the sigma notation to write (do not evaluate): $\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} = \sum_{k=3}^{10}\frac{(-1)^{k-1}}{k}$

c) Find $\int_4^{-1} f(x)\,dx$ if $\int_{-1}^{1} f(x)\,dx = -3$ and $\int_1^4 f(x)\,dx = 7$.

$$\int_4^{-1} f(x)\,dx = \int_4^1 f(x)\,dx + \int_1^{-1} f(x)\,dx = -\int_1^4 f(x)\,dx - \int_{-1}^1 f(x)\,dx = -7 - (-3) = -4$$

d) State the Fundamental Theorem of Calculus, part 1.

If f is continuous on the interval $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

3. [10] State whether the statement is true or false. No explanation is needed.

a) If f is integrable on $[-2,3]$, then f is continuous on $[-2,3]$. False

b) $\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big]_{-1}^{1} = -(1-(-1)) = -2.$ False ; FTC1 does not apply, as $f(x) = \frac{1}{x^2}$ is not continuous on $[-1,1]$

c) If f is continuous on $[0,3]$, then f is integrable on $[0,3]$. True, by Theorem 5.5.2

d) If $\int_1^3 f(x)\,dx = 4$, then $\int_1^2 f(2x-1)\,dx = 2$. True; set $u = 2x-1$; $du = 2dx$. $\int_1^2 f(2x-1)dx = \int_1^3 f(u)\frac{du}{2}$
$\frac{1}{2}\int_1^3 f(u)du = \frac{1}{2}(4) = 2.$

e) We may use the change of variable $u = x$ to evaluate $\int \frac{x^4}{(1-x^2)^{\frac{5}{2}}}\,dx$. False, as $u = x$ does not simplify the integration process.

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4. [20] Computation of integrals; Show all your work.

a) $\int_0^1 \ln(\sqrt{x+1})\,dx = \frac{1}{2}\int_0^1 \ln(x+1)dx = \frac{1}{2}(x+1)\ln(x+1)\Big]_0^1 - \frac{1}{2}\int_0^1 (x+1)\frac{dx}{x+1}$

$= \frac{1}{2}(2)\ln(2) - \frac{1}{2}\ln(1) - \frac{x}{2}\Big]_0^1$

$= \ln(2) - \frac{1}{2}$, as $\ln(1) = 0$.

$u = \ln(x+1)$
$dv = dx$
$du = \frac{dx}{x+1}$
$v = x+1$

This choice of v simplifies computations (you may choose $v = x$ as well)

b) $\int_{-2}^{1} \sqrt{2-|x|}\,dx = \int_{-2}^{0}\sqrt{2-(-x)}\,dx + \int_0^1 \sqrt{2-x}\,dx = \int_0^2 \sqrt{u}\,du + \int_2^1 \sqrt{v}\,(-dv)$

$= \frac{2}{3}u^{3/2}\Big]_0^2 + \frac{2}{3}v^{3/2}\Big]_1^2$

$= \frac{2}{3}(2\sqrt{2}) + \frac{2}{3}(2\sqrt{2}-1)$

$= \frac{8}{3}\sqrt{2} - \frac{2}{3}$

$|x| = 0 \rightarrow x = 0$
$u = 2+x$; $du = dx$
$v = 2-x$; $dv = -dx$

c) $\int \sec^9 x \tan^3 x\,dx = \int \sec^8 x \tan^2 x \sec x \tan x\,dx = \int u^8(u^2-1)du$

$= \frac{u^{11}}{11} - \frac{u^9}{9} + C$

$= \frac{\sec^{11} x}{11} - \frac{\sec^9 x}{9} + C$

$u = \sec x$
$du = \sec x \tan x\,dx$

d) If $f(x) = \frac{x^3+3x}{x^2+1}$, find the average value of f on $[-2,1]$.

$f_{ave} = \frac{1}{3}\int_{-2}^{1} \frac{x(x^2+1)}{x^2+1} + \frac{2x}{x^2+1}\,dx = \frac{1}{3}\int_{-2}^{1} x\,dx + \frac{1}{3}\int_{-2}^{1}\frac{2x}{x^2+1}dx$

$= \frac{1}{3}\frac{x^2}{2}\Big]_{-2}^{+1} + \frac{1}{3}\int_5^2 \frac{du}{u}$

$= \frac{1}{6}(1-4) + \frac{1}{3}\ln u\Big]_5^2$

$= -\frac{1}{2} + \frac{1}{3}(\ln 2 - \ln 5)$

$u = x^2+1$
$du = 2x\,dx$

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5. [7] If $F(x) = \int_1^{\cos(2x)} \frac{t^3+t^2+t^1+1}{t^4+2}\,dt$, use part 2 of the fundamental theorem of Calculus to find its derivative $F'(x)$. What is $F(0)$? $F'(0)$?

Set $h(u) = \int_1^u \frac{t^3+t^2+t+1}{t^4+2}\,dt$. $F(x) = h(\cos(2x))$. $h'(u) = \frac{u^3+u^2+u+1}{u^4+2}$ by FTC2

$F'(x) = -2\sin(2x)\cdot h'(\cos(2x)) = -2\sin(2x)\cdot\left(\frac{\cos^3(2x)+\cos^2(2x)+\cos(2x)+1}{\cos^4(2x)+2}\right)$

$F(0) = \int_1^1 \frac{t^3+t^2+t+1}{t^4+2}dt = 0$; $F'(0) = -2\sin(0)\left(\frac{1+1+1+1}{1+2}\right) = 0$, as $\sin(0) = 0$

6. [10] a) Given that $a = 0$ and $b = 1$, use those values to express the following limit as an integral, but do not evaluate the integral:

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \sec^3\left(\frac{k\pi}{n}\right) = \int_0^1 \sec^3(\pi x)\,dx, \quad \text{as} \quad x_k^* = x_k = \frac{k}{n},\ \Delta x = \frac{1}{n}$$

b) Write the integral $\int_0^{\pi} \sin(2+3^x)\,dx$ as the limit of a Riemann sum, but do not evaluate the integral.

$$\int_0^{\pi} \sin(2+3^x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sin(2+3^{x_k^*})\Delta x_k; \quad a=0,\ b=\pi$$

or

$$\int_0^{\pi} \sin(2+3^x)\,dx = \lim_{n\to\infty} \frac{\pi}{n}\sum_{k=1}^{n} \sin\left(2+3^{\frac{k\pi}{n}}\right), \quad x_k^* = x_k = \frac{k\pi}{n},\ \Delta x = \frac{\pi}{n}$$

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7. [13] Evaluate each integral:

a) $\int \cos(2x)\cos(6x)\,dx = \int \frac{\cos(6x+2x)+\cos(6x-2x)}{2}dx$

$$= \frac{1}{2}\int(\cos(8x)+\cos(4x))dx$$

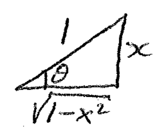
$$= \frac{1}{2}\left(\frac{\sin(8x)}{8}+\frac{\sin(4x)}{4}\right)+C$$

b) $\int \frac{x^4}{(1-x^2)^{\frac{5}{2}}}dx = \int \frac{\sin^4\theta\cos\theta\,d\theta}{(\cos^2\theta)^{5/2}} = \int \frac{\sin^4\theta\cos\theta}{\cos^5\theta}d\theta = \int \tan^4\theta\,d\theta$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$$= \int \tan^2\theta(\sec^2\theta-1)d\theta, \quad \text{as } \tan^2\theta = \sec^2\theta - 1$$



$$= \int \tan^2\theta\sec^2\theta\,d\theta - \int(\sec^2\theta-1)d\theta$$

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$$= \int u^2du - \tan\theta + \theta$$

$$= \frac{u^3}{3} - \tan\theta + \theta + C = \frac{\tan^3\theta}{3} - \tan\theta + \theta + C$$

$$= \frac{x^3}{3(1-x^2)^{3/2}} - \frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x) + C$$

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8. [**Bonus**, 10] A particle is moving on a straight line with acceleration $a(t) = \cos(2t)$ m/s^2, $t \geq 0$. Find its displacement and distance traveled on the interval $[0, \frac{\pi}{2}]$, given that the initial velocity est $v(0) = 0$.

$$v(t) = \int a(t)dt = \int\cos(2t)dt = \frac{\sin(2t)}{2} + C$$

$$v(0) = 0 + C = 0 \to C = 0$$

$v(t) = \frac{\sin(2t)}{2}$ since $0 \le 2t \le \pi$, $v(t) \geq 0$ on $[0, \pi/2]$.

Hence displacement = distance traveled $= \int_0^{\pi/2} \frac{\sin(2t)}{2}dt = -\frac{\cos(2t)}{4}\Big]_0^{\pi/2}$

$$= \frac{-1}{4}(\cos(\pi) - \cos(0))$$

$$= -\frac{1}{4}(-2) = \frac{1}{2}.$$