MAC 2313 (Calculus III) - Key
Test 1, Monday October 14, 2013

Name:

PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve any of the points assigned to any question. *You will not get any credit if you do not show the steps to your answers.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. 3 pages. Total=100 points. Good luck.**

1. [6] Describe the surface $x^2+y^2+z^2+6x-2y-6=0$. If it is a sphere, state its radius and center. If it is a point, state its coordinates.

$z^2+x^2+6x+9+y^2-2y+1=6+9+1$

$(x+3)^2+(y-1)^2+z^2=16$

Sphere, center $=(-3,1,0)$, radius $=4$

2. [12] a) Find the volume of the parallelepiped having $\vec{u}=2\vec{i}-3\vec{j}+7\vec{k}$, $\vec{v}=\vec{i}-2\vec{j}+\vec{k}$, and $\vec{w}=\vec{i}-3\vec{j}-\vec{k}$ as adjacent edges.

$V=|\vec{u}\cdot(\vec{v}\times\vec{w})|$. Now $\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix}2&-3&7\\1&-2&1\\1&-3&-1\end{vmatrix}=2(2+3)-(-3)(-1-1)+7(-3+2)=10-6-7=-3$

$=|-3|$

$=3$

b) Find the area of the face determined by $\vec{u}$ and $\vec{v}$.

$A=\|\vec{u}\times\vec{v}\|$. Now $\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-3&7\\1&-2&1\end{vmatrix}=(-3+14)\vec{i}-(2-7)\vec{j}+(-4+3)\vec{k}=11\vec{i}+5\vec{j}-\vec{k}$

$=\sqrt{11^2+5^2+1}$

$=\sqrt{147}$

c) Find the angle θ between $\vec{w}$ and the plane determined by $\vec{u}$ and $\vec{v}$. $\vec{u}\times\vec{v}=$ normal to plane.

$\cos\theta=\dfrac{\vec{w}\cdot(\vec{u}\times\vec{v})}{\|\vec{w}\|\,\|\vec{u}\times\vec{v}\|}=\dfrac{\vec{u}\cdot(\vec{v}\times\vec{w})}{\sqrt{11}\sqrt{147}}=\dfrac{-3}{\sqrt{11}\sqrt{147}}$, $\theta=\cos^{-1}\left(\dfrac{-3}{\sqrt{11}\sqrt{147}}\right)$.

3. [6] a) Let $A(x_0,y_0,z_0)$ be a given point in 3-space. Let $\mathcal{P}$ be the plane with equation $ax+by+cz+d=0$. Write down the distance D between A and the plane $\mathcal{P}$.

$D=\dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$

b) Use a) to find the distance between the two parallel planes: $\mathcal{P}_1: 2x+3y-z=4$ and $\mathcal{P}_2: 2x+3y-z=-3$.

The point $A(2,0,0)$ lies on $\mathcal{P}_1$. $d(\mathcal{P}_1,\mathcal{P}_2)=d(A,\mathcal{P}_2)$

Now $d(A,\mathcal{P}_2)=\dfrac{|2(2)+3(0)-1(0)+3|}{\sqrt{4+9+1}}=\dfrac{7}{\sqrt{14}}$

So the distance between $\mathcal{P}_1$ & $\mathcal{P}_2$ is $\dfrac{7}{\sqrt{14}}$ or $\dfrac{\sqrt{14}}{2}$.

4. [6] Let $\vec{w}=\vec{i}-2\vec{j}+3\vec{k}$ and $\vec{v}=2\vec{i}-\vec{j}-\vec{k}$. Find the vector component of $\vec{v}$ that is parallel to $\vec{w}$ and the vector component of $\vec{v}$ that is orthogonal to $\vec{w}$.

Vector component of $\vec{v}$ parallel to $\vec{w}=\operatorname{Proj}_{\vec{w}}(\vec{v})=\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\vec{w}=\dfrac{1}{14}\vec{w}$

$\vec{v}\cdot\vec{w}=2+2-3=1$

$\|\vec{w}\|^2=1+4+9=14.$

Vector component of $\vec{v}$ orthogonal to $\vec{w}=\vec{v}-\operatorname{Proj}_{\vec{w}}(\vec{v})$

$=-\dfrac{(\vec{i}-2\vec{j}+3\vec{k})}{14}+2\vec{i}-\vec{j}-\vec{k}$

$=\left(2-\frac{1}{14}\right)\vec{i}+\left(-1+\frac{1}{7}\right)\vec{j}-\left(1+\frac{3}{14}\right)\vec{k}$

5. [10] Let $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

a) Find the partial derivatives $f_x(0,0)$, and $f_y(0,0)$. b) Is f differentiable at $(0,0)$?

a) $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$, $f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0}{k} = 0$

b) $\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$, set $h = r\cos\theta$, $k = r\sin\theta$, $r>0, 0\le\theta<2\pi$

$= \lim_{(h,k)\to(0,0)} \frac{h^2k}{(h^2+k^2)^{3/2}} = \lim_{r\to 0^+} \frac{r^3\cos^2\theta\sin\theta}{r^3}$

$= \cos^2\theta\sin\theta = \begin{cases} 0 & \text{if } \theta = 0 \\ \frac{1}{2\sqrt{2}} & \text{if } \theta = \frac{\pi}{4} \end{cases}$

Since limit $\neq 0$, and worse DNE, f is not differentiable at $(0,0)$.

6. [12] If $3x = u^3+v^3$ and $2y = u^2+v^2$, set $z = e^{uv}$. Use implicit partial differentiation to find u_x and v_x. Then find z_x.

$\frac{d}{dx}(3x) = \frac{\partial}{\partial x}(u^3+v^3)$

$\cancel{3}1 = \cancel{3}u_xu^2+\cancel{3}v_xv^2$ (1)

$\frac{\partial}{\partial x}(2y) = \frac{\partial}{\partial x}(u^2+v^2)$

$0 = \cancel{2}u_xu+\cancel{2}v_xv$ (2)

$(1)-v\cdot(2)$: $1 = u_x(u^2-uv) \to u_x = \frac{1}{u^2-uv}$

$(1)-u\cdot(2)$: $1 = v_x(v^2-uv) \to v_x = \frac{1}{v^2-uv}$,

$z_x = z_uu_x+z_vv_x$

$z_u = ve^{uv}$, $z_v = ue^{uv}$

$z_x = \left(\frac{v}{u^2-uv}+\frac{u}{v^2-uv}\right)e^{uv}$

7. [10] Show that the two lines $L_1: x = 1-t,\ \ y = 2+t,\ \ z = 1+5t$, and $L_2: x = 2+t,\ \ y = 2+3t,\ \ z = -1+7t$ intersect, and find their point of intersection A.

Do there exist t_1 and t_2 such that

$1-t_1 = 2+t_2$ (1)

$2+t_1 = 2+3t_2$ (2) $\to t_1 = 3t_2$ (4)

$1+5t_1 = -1+7t_2$ (3)

(4) in (1): $1-3t_2 = 2+t_2 \to 4t_2 = 1-2 = -1$

$t_2 = -1/4$ (5)

(5) in (4): $t_1 = -3/4$ (6)

Now (5) and (6) in (3):

LHS $= 1-\frac{15}{4} = -\frac{11}{4} = -1-\frac{7}{4} =$ RHS;

So L_1 & L_2 intersect at $A\left(\frac{7}{4}, \frac{5}{4}, -\frac{11}{4}\right)$.

8. [14] Decide whether the statement is true or false. No explanation is needed.

a) If f is differentiable at $(1,-1,2)$, then f is continuous at $(1,-1,2)$. True

b) If $\lim_{(x,y)\to(-1,1)} f(x,y) = 3$, then $f(x,y)\to 3$ as (x,y) approaches $(-1,1)$ along the line $y=1$ and $f(x,y)\to 3$ as (x,y) approaches $(-1,1)$ along the parabola $y = 1+(x+1)^2$. True

c) If $z(t) = f(x(t),y(t))$, then $\frac{dz}{dt} = f_x(\frac{dx}{dt},y)+f_y(x,\frac{dy}{dt})$. False, $z'(t) = f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t)$

d) The equation $y = x^2$ represents a parabola in 3-space. False, a parabolic cylinder

e) The vector $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. True

f) If $f(x,y) = \ln(x^2-y^2)$, then f is continuous on $\{(x,y); x^2-y^2>0\}$. True

g) If $f = f(x,y,z)$ is differentiable at the point $A(1,2,3)$, then the directional derivative of f at A in the direction of the vector $\vec{b} = \vec{i}+2\vec{j}+3\vec{k}$ is given by $D_{\vec{b}}f(A) = \nabla f(A)\cdot\vec{b}$. False, $D_{\vec{b}}f(A) = \nabla f(A)\cdot\frac{\vec{b}}{\|\vec{b}\|}$

9 . [24] a) Find an equation for the tangent plane and parametric equations for the normal line to the paraboloid $z = 4 - x^2 - y^2$ at the point (1,-1,2).

Set $F(x,y,z) = z - 4 + x^2 + y^2$. $\nabla F(x,y,z) = \langle 2x, 2y, 1 \rangle$

$\nabla F(1,-1,2) = \langle 2,-2,1 \rangle$ = a normal to tangent plane

Equation of tangent plane : $2(x-1) - 2(y+1) + z - 2 = 0$

Parametric equations of normal line

$x = 1 + 2t, \quad y = -1 - 2t, \quad z = 2 + t$

b) Let $f(x,y) = 2x^2 - 4xy + y^4 + 2$. Find all the critical points of f and classify each of them as a local maximum, a local minimum, or a saddle point.

$f_x(x,y) = 4x - 4y$, $f_y(x,y) = -4x + 4y^3$, $\nabla f(x,y) = \vec{0} \Rightarrow x = y$ & $x = y^3$

so $y - y^3 = 0$ or $y(y^2 - 1) = 0$ or $y(y-1)(y+1) = 0$. Hence $y = 0$, $y = 1$ or $y = -1$

CPs: $(0,0)$, $(1,1)$, and $(-1,-1)$.

$f_{xx}(x,y) = 4$, $f_{xy}(x,y) = -4$, $f_{yy}(x,y) = 12y^2$

$\Delta(x_0, y_0) = f_{xy}(x_0,y_0)^2 - f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) = 16 - 48y^2$

(x_0, y_0)	$(0,0)$	$(1,1)$	$(-1,-1)$
$\Delta(x_0,y_0)$	16	-32	-32
$f_{xx}(x_0,y_0)$		4	4
Classification	Saddle point	local minimum	local minimum