MAC 2313 (Calculus III) — Answers
Test 1, Wednesday September 16, 2015

Name: PID:

**Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may use the back of the page to show your work. Total=65 points.**

1 [5] Find an equation for the sphere having the points $A(1,2,3)$ and $B(3,4,5)$ as endpoints of a diameter.

If $r$ = radius of sphere, then $r=\frac{1}{2}d(A,B)=\frac{1}{2}\sqrt{(3-1)^2+(4-2)^2+(5-3)^2}=\frac{1}{2}\sqrt{12}=\sqrt{3}$

$C$ = center of sphere = midpoint of $A$ & $B$ $=\left(\frac{1+3}{2},\frac{4+2}{2},\frac{5+3}{2}\right)=(2,3,4)$

Eqn: $(x-2)^2+(y-3)^2+(z-4)^2=3$

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2. [10] a) Find parametric equations for the line of intersection $L$ of the two planes: $\mathcal{P}_1: 2x-y+3z=4$ and $\mathcal{P}_2: x+y-z=5$.

$\vec{n}_1=\langle 2,-1,3\rangle$ = normal to $\mathcal{P}_1$

$\vec{n}_2=\langle 1,1,-1\rangle$ = normal to $\mathcal{P}_2$

$\vec{u}=\vec{n}_1\times\vec{n}_2$ = vector parallel to $L$.

$\vec{u}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & 3\\ 1 & 1 & -1\end{vmatrix}=(1-3)\vec{i}-(-2-3)\vec{j}+(2+1)\vec{k}=-2\vec{i}+5\vec{j}+3\vec{k}$

It remains to find a point on $L$; to do this, you may set $z=0$ in equations of $\mathcal{P}_1$ and $\mathcal{P}_2$ and solve for $x$ and $y$:

$2x-y=4$ (1)

$x+y=5$ (2)

(1)+(2) yield: $3x=9$ or $x=3$. (2) yields $y=5-x=2$ so $A(3,2,0)$ lies on $L$. Parametric equations of $L$: $x=3-2t$, $y=2+5t$, $z=3t$

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3. [12] Let $\vec{w}=\vec{i}-2\vec{j}-3\vec{k}$, and $\vec{z}=3\vec{j}-2\vec{k}$. a) Find the vector component of $\vec{w}$ that is parallel to $\vec{z}$.

Vector component of $\vec{w}$ parallel to $\vec{z}$ = $\text{proj}_{\vec{z}}(\vec{w})=\frac{\vec{w}\cdot\vec{z}}{\|\vec{z}\|^2}\vec{z}=\frac{(-6+6)\vec{z}}{9+4}=\vec{0}$ since $\vec{w}$ and $\vec{z}$ are orthogonal

b) If $\theta$ is the angle between $\vec{w}$ and $\vec{z}$, find $\cos(\theta)$ and $\sin(\theta)$.

Since $\vec{w}$ and $\vec{z}$ are orthogonal, $\theta=\frac{\pi}{2}$ radians or $\theta=90°$; so $\cos\theta=0$ and $\sin\theta=1$.

c) If a force $\vec{F}=\vec{z}$ is applied to move an object 12 meters in the direction of the vector $\vec{w}$, find the work done by $\vec{F}$.

$W=\vec{F}\cdot\frac{\vec{w}}{\|\vec{w}\|}(12)=\frac{\vec{z}\cdot\vec{w}}{\sqrt{1+4+9}}(12)=0$ since $\vec{z}\cdot\vec{w}=0$

No work is done.

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4. [8] Set $\vec{u}=\vec{i}+\vec{j}-\vec{k}$, $\vec{v}=-2\vec{i}+\vec{k}$ and $\vec{r}=\vec{j}-\vec{k}$. a) Find the area of the parallelogram having $\vec{u}$ and $\vec{r}$ as adjacent sides. b) Find the volume of the parallelepiped having $\vec{u}$, $\vec{v}$ and $\vec{r}$ as adjacent edges.

$A=\|\vec{r}\times\vec{u}\|$. $\vec{r}\times\vec{u}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 1 & -1\\ 1 & 1 & -1\end{vmatrix}=(-1+1)\vec{i}-(0+1)\vec{j}+(0-1)\vec{k}=-\vec{j}-\vec{k}$

$=\sqrt{2}$

b) $V=|\vec{v}\cdot(\vec{r}\times\vec{u})|=|-2(0)+0(-1)+1(-1)|=1$

5. [20] a) Show that the two lines $L_1 : x = -1+t, \quad y = -2+t, \quad z = -2+2t$, and $L_2 : x = 5+2t, \quad y = 3+t, \quad z = 5-t$ intersect, and find their point of intersection $A$.
b) Find an equation for the plane $\mathcal{P}$ that contains both $L_1$ and the point $C(1, 2, -1)$.
c) Find the distance between the plane $\mathcal{P}$ and the point $B(1, 1, 1)$.

a) Do there exist $t_1$ and $t_2$ with:

$-1+t_1 = 5+2t_2 \quad (1)$

$-2+t_1 = 3+t_2 \quad (2)$

$-2+2t_1 = 5-t_2 \quad (3)$

$(2)+(3)$ yield: $3t_1 - 4 = 8$; so $t_1 = \frac{12}{3} = 4 \quad (4)$

$(4)$ in $(2)$ yields $t_2 = -2+4-3 = -1 \quad (5)$

Do $t_1$ & $t_2$ satisfy $(1)$? $LHS = -1+4 = 3 = 5+2(-1) = RHS$

So $L_1$ and $L_2$ intersect at $A(3, 2, 6)$.

b) $\vec{u_1} = \langle 1, 1, 2 \rangle \parallel L_1$. $\vec{AC} = \langle 1-3, 2-2, -1-6 \rangle = \langle -2, 0, -7 \rangle$

$\vec{n} = \vec{u_1} \times \vec{AC}$ = normal to $\mathcal{P}$ = $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ -2 & 0 & -7 \end{vmatrix} = (-7-0)\vec{i} - (-7+4)\vec{j} + (0+2)\vec{k} = -7\vec{i} + 3\vec{j} + 2\vec{k}$

$A$ lies on $\mathcal{P}$ (You may choose $C$ if you want to).

Eqn for $\mathcal{P}$: $-7(x-3)+3(y-2)+2(z-6) = 0$ or $-7x+3y+2z+3=0$

c) $d(B, \mathcal{P}) = \dfrac{|-7(1)+3(1)+2(1)+3|}{\sqrt{7^2+9+4}} = \dfrac{1}{\sqrt{62}}$

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6. [4] Find an equation for the surface that results when the elliptic cone $x = \sqrt{2y^2 + 4z^2}$ is reflected about the plane: i) $x = 0$, ii) $x = y$.

i) $x = -\sqrt{2y^2+4z^2}$

ii) $y = \sqrt{2x^2+4z^2}$

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7 [6]. a) Convert the point $(-2, -2, 4)$ from rectangular to spherical coordinates. b) Convert the equation $r^2 + \frac{z^2}{4} = 4$ from cylindrical to rectangular coordinates, and identify the surface.

a) $\rho = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$

$\left.\begin{array}{l} \tan\theta = \frac{-2}{-2} = 1 \\ \theta \text{ in } Q\,III \end{array}\right\}$ so $\theta = \frac{5\pi}{4}$

$\phi = \cos^{-1}\left(\frac{4}{2\sqrt{6}}\right) = \cos^{-1}\left(\frac{\sqrt{6}}{3}\right)$

$\left(2\sqrt{6}, \frac{5\pi}{4}, \cos^{-1}\left(\frac{\sqrt{6}}{3}\right)\right)$

b) $x^2+y^2+\frac{z^4}{4} = 4$

$\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} = 1$, Ellipsoid