MAC 2313 (Calculus III)
Test 1, Friday October 6, 2017 — Answers

Name:

PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work. Guessing the correct answers won't give you any credits. You must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may use the back of the page to show your work. Total= 90 points. Always do your best.

1. [10] Write down an equation for the plane $\mathcal{P}$ that contains the three points $A(-1,2,3)$, $B(1,1,1)$ and $C(2,3,1)$, and parametric equations for the line L that contains the points A and C.

$\vec{AB} = \langle 1+1, 1-2, 1-3\rangle = \langle 2, -1, -2\rangle$; $\vec{AC} = \langle 2+1, 3-2, 1-3\rangle = \langle 3, 1, -2\rangle$

$\vec{n} = \vec{AB}\times\vec{AC}$ = a normal to plane $\mathcal{P}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = (2+2)\vec{i} - (-4+6)\vec{j} + (2+3)\vec{k} = 4\vec{i} - 2\vec{j} + 5\vec{k}$$

Equation for $\mathcal{P}$: $4(x+1) - 2(y-2) + 5(z-3) = 0$, using A.

$\vec{AC}$ is parallel to L. So, parametric equations for L are:

$x = 2 + 3t$, $y = 3 + t$, $z = 1 - 2t$, $-\infty < t < +\infty$

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2. [15] a) Write down an equation for the sphere that has the points A and B from problem 1 as endpoints of a diameter.

Center = midpoint of A and B = $\left(\frac{-1+1}{2}, \frac{2+1}{2}, \frac{3+1}{2}\right) = \left(0, \frac{3}{2}, 2\right)$

radius $= \frac{1}{2} d(A,B) = \frac{1}{2}\|\vec{AB}\| = \frac{1}{2}\sqrt{4+1+4} = \frac{3}{2}$

Equation for Sphere: $x^2 + \left(y - \frac{3}{2}\right)^2 + (z-2)^2 = \frac{9}{4}$

b) Convert the equation $2\sin(\theta) = 1$ from cylindrical to rectangular coordinates, and identify the surface.

$2r\sin\theta = r$; so $(2r\sin\theta)^2 = r^2$ or $4y^2 = x^2 + y^2$

So $x^2 - 3y^2 = 0$ or $(x - \sqrt{3}y)(x + \sqrt{3}y) = 0$

Hence $\underbrace{x - \sqrt{3}y = 0}_{\text{Plane}}$ or $\underbrace{x + \sqrt{3}y = 0}_{\text{Plane}}$

So surface is the union of the two planes $x - \sqrt{3}y = 0$ and $x + \sqrt{3}y = 0$.

c) Convert the point $(-2,-2,2)$ from rectangular to spherical coordinates.

$\rho = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$

$\tan\theta = \frac{-2}{-2} = 1$, θ in Q3 $\rightarrow \theta = \frac{5\pi}{4}$

$\cos\phi = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \rightarrow \phi = \cos^{-1}(1/\sqrt{3})$

Hence $\left(2\sqrt{3}, \frac{5\pi}{4}, \cos^{-1}(1/\sqrt{3})\right)$

3. [17] Let $\vec{p} = 2\vec{i} - \vec{j} - 2\vec{k}$, and $\vec{q} = 3\vec{i} + \vec{j} - 2\vec{k}$. a) Find the vector component of $\vec{q}$ that is parallel to $\vec{p}$.

$$Proj_{\vec{p}}(\vec{q}) = \frac{\vec{p}\cdot\vec{q}}{\|\vec{p}\|^2}\vec{p} = \frac{2(3)-(1)-2(-2)}{4+1+4}\vec{p} = \vec{p}$$

$$\vec{p}\times\vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= (2+2)\vec{i} - (-4+6)\vec{j} + (2+3)\vec{k}$$

$$= 4\vec{i} - 2\vec{j} + 5\vec{k}$$

b) If θ is the angle between $\vec{p}$ and $\vec{q}$, find $\cos(\theta)$ and $\sin(\theta)$. Is θ acute or obtuse?

$$\cos\theta = \frac{\vec{p}\cdot\vec{q}}{\|\vec{p}\|\|\vec{q}\|} = \frac{9}{3\sqrt{9+1+4}} = \frac{9}{3\sqrt{14}} = \frac{3}{\sqrt{14}}$$

$$\sin\theta = \frac{\|\vec{p}\times\vec{q}\|}{\|\vec{p}\|\|\vec{q}\|} = \frac{\sqrt{16+4+25}}{3\sqrt{14}} = \frac{\sqrt{45}}{\sqrt{14}} = \frac{\sqrt{5}}{\sqrt{14}}.$$

θ is acute since $\cos\theta > 0$.

c) Find the area of the parallelogram having $\vec{p}$ and $\vec{q}$ as adjacent edges.

$$A = \|\vec{p}\times\vec{q}\| = \sqrt{45} = 3\sqrt{5}$$

d) Set $\vec{r} = \vec{i} - \vec{j} - 2\vec{k}$. Find the volume of the parallelepiped having $\vec{p}$, $\vec{q}$ and $\vec{r}$ as adjacent edges.

$$V = |\vec{r}\cdot(\vec{p}\times\vec{q})| = |1(4) - (-2) - 2(5)| = |-4| = 4$$

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4. [15] a) Show that the line $L : x = 1 + 2t,\ y = -1 - t,\ z = 3 - 2t$ is parallel to the plane $\mathcal{P} : 2x + 2y + z = 7$, and find the distance between the line and the plane.

$\mathcal{P}: 2x+2y+z-7=0$

$\vec{u} = \langle 2, -1, -2\rangle \parallel L$

$\vec{n} = \langle 2, 2, 1\rangle$ = a normal to $\mathcal{P}$

L is parallel to $\mathcal{P}$ if $\vec{u}\cdot\vec{n} = 0$. Now $\vec{u}\cdot\vec{n} = 2(2) - 2 - 2 = 0$; so L is parallel to $\mathcal{P}$. Since L is parallel to $\mathcal{P}$, $d(L,\mathcal{P}) = d(D,\mathcal{P})$ for any point D on L. Pick $D = (1,-1,3)$, just set $t=0$.

$$d(L,\mathcal{P}) = d(D,\mathcal{P}) = \frac{|2(1)+2(-1)+1(3)-7|}{\sqrt{4+4+1}} = \frac{|-4|}{3} = \frac{4}{3}$$

b) Write down an equation for the surface that results when the hyperboloid of one sheet $x^2 - 9y^2 + 4z^2 = 1$ is reflected about the plane $y = z$. Also identify the surface.

$x^2 + 4y^2 - 9z^2 = 1$, hyperboloid of one sheet, along z-axis

just switch y & z.

$\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$
$= \|\vec{u}\|^2 + 0 + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$, as $\vec{u}$ and $\vec{v}$ are orthogonal.

5. [10] Decide whether the statement is true or false. No explanation needed.

a) If $\vec{u}$ and $\vec{v}$ are any two vectors, then $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. True

$\vec{u}\cdot(\vec{u}\times\vec{v}) = (\vec{u}\times\vec{u})\cdot\vec{v} = \vec{0}\cdot\vec{v} = 0$

$(\vec{u}\times\vec{v})\cdot\vec{v} = (\vec{v}\times\vec{v})\cdot\vec{u} = \vec{0}\cdot\vec{u} = 0$

b) If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$. True

c) If f is differentiable at $(1,-1,2)$, then f is continuous at $(1,-1,2)$. True, Theorem 13.4.3

d) If $\lim_{(x,y)\to(-1,1)} f(x,y) = 3$, then $f(x,y)\to 3$ as (x,y) approaches $(-1,1)$ along the line $y=1$ and $f(x,y)\to 3$ as (x,y) approaches $(-1,1)$ along the parabola $y = 1+(x+1)^2$. True, Theorem 13.2.2

e) The equation $y = x^2$ represents a parabola in 3-space. False, parabolic cylinder

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6. [10, **Bonus**] Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $y^3 + z^2x + \tan(zy) = -8$.

$\frac{\partial}{\partial x}\left(y^3 + z^2x + \tan(zy)\right) = \frac{\partial}{\partial x}(-8) = 0$

$0 + 2xz\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x}\sec^2(zy) + z^2 = 0$; $\frac{\partial z}{\partial x}\left(2xz + y\sec^2(yz)\right) = -z^2$

$\frac{\partial z}{\partial x} = -z^2/\left(2xz + y\sec^2(yz)\right)$

$\frac{\partial}{\partial y}\left(y^3 + z^2x + \tan(zy)\right) = \frac{\partial}{\partial y}(-8) = 0$; $3y^2 + 2xz\frac{\partial z}{\partial y} + \left(y\frac{\partial z}{\partial y} + z\right)\sec^2(yz) = 0$

so $\left(2xz + y\sec^2(yz)\right)\frac{\partial z}{\partial y} = -3y^2 - z\sec^2(yz)$; hence

$\partial z/\partial y = -\left(3y^2 + z\sec^2(yz)\right)/\left(2xz + y\sec^2(yz)\right)$

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7. [13] Let $f(x,y,z) = x^5y^2e^{2xz}$. a) Find the partial derivatives f_x, f_y and f_z. b) Find $f(-2y, 3z, x^2)$. c) Find the local linear approximation L to f at the point $A(0,1,-1)$. d) Use the local linear approximation L to f at A to approximate $f(0.03, 0.99, -1.02)$.

a) $f_x(x,y,z) = 5x^4y^2e^{2xz} + 2x^5y^2z\,e^{2xz}$

$f_y(x,y,z) = 2x^5y\,e^{2xz}$; $f_z(x,y,z) = 2x^6y^2e^{2xz}$

b) $f(-2y, 3z, x^2) = (-2y)^5(3z)^2e^{2(-2y)x^2} = -32(9)y^5z^2e^{-4x^2y}$

c) $L(x,y,z) = f(A) + f_x(A)x + f_y(A)(y-1) + f_z(A)(z+1)$

$= 0 + 0.x + 0.(y-1) + 0.(z+1)$

$f(0.03, 0.99, -1.02) \approx L(0.03, 0.99, -1.02) = 0.$