MAC 2313 (Calculus III) — *Answers*
Test 1, Thursday May 25, 2017

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work. Guessing the correct answers won't give you any credits. You must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may use the back of the page to show your work. Total= 90 points.

1. [10] a) Find the distance between the two parallel planes: $\mathcal{P}_1 : 2x - 3y + z = 5$ and $\mathcal{P}_2 : 2x - 3y + z = 8$.

Select any point A on $\mathcal{P}_1$. Then $d(\mathcal{P}_1, \mathcal{P}_2) = d(A, \mathcal{P}_2)$. Setting $x=0=y$, we easily get $A(0,0,5)$ on $\mathcal{P}_1$. Using the distance formula, we find $d(A, \mathcal{P}_2) = \dfrac{|2(0)-3(0)+1(5)-8|}{\sqrt{4+9+1}} = \dfrac{3}{\sqrt{14}}$.

b) Show that the line $L : x = -3 + t, \quad y = 5 + 2t, \quad z = -1 + t$ is parallel to the plane $Q : 2x - 3y + 4z = -1$.

The plane Q and line L are parallel if $\vec{u} = \langle 1,2,1\rangle$, vector parallel to L and $\vec{n} = \langle 2,-3,4\rangle$, vector normal to Q are orthogonal. Now $\vec{u}\cdot\vec{n} = 1(2)+2(-3)+1(4) = 2-6+4 = 0$; hence L is parallel to Q.

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2. [10] a) If a point A is given in spherical coordinates as (ρ, θ, ϕ), write down the rectangular coordinates (x, y, z) of A.

$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$

b) Convert the equation $\rho = -2\sin(\phi)\sin(\theta)$ from spherical to rectangular coordinates, and identify the surface.

Multiply both sides of the equation by ρ to get: $\rho^2 = -2\rho\sin\phi\sin\theta$, or
$x^2+y^2+z^2 = -2y$ or $x^2+y^2+2y+1+z^2 = 1$ or
$x^2+(y+1)^2+z^2 = 1$; Sphere, Center $= (0,-1,0)$, radius $= 1$

c) Convert the point $(-7, 7, 5)$ from rectangular to cylindrical coordinates.

$r = \sqrt{49+49} = \sqrt{98} = 7\sqrt{2}$. $\left.\begin{array}{l}\tan\theta = \frac{7}{-7} = -1\\ \theta \text{ in } Q2\end{array}\right\} \to \theta = \frac{3\pi}{4}$, $z = 5$

Hence $(7\sqrt{2}, \frac{3\pi}{4}, 5)$ in cylindrical coordinates

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3. [25] Let $\vec{r} = 2\vec{i} + \vec{j} + 5\vec{k}$, and $\vec{s} = -\vec{i} + 3\vec{j} - 4\vec{k}$. a) Find the vector component of $\vec{s}$ that is parallel to $\vec{r}$.

vector component of $\vec{s}$ parallel to $\vec{r} = \mathrm{Proj}_{\vec{r}}(\vec{s}) = \dfrac{\vec{s}\cdot\vec{r}}{\|\vec{r}\|^2}\vec{r} = \dfrac{-19}{30}(2\vec{i}+\vec{j}+5\vec{k})$
Now $\vec{s}\cdot\vec{r} = -1(2)+3(1)-4(5) = -19$
$\|\vec{r}\|^2 = 4+1+25 = 30$

b) If θ is the angle between $\vec{r}$ and $\vec{s}$, find $\cos(\theta)$ and $\sin(\theta)$. Is θ acute or obtuse?

$\cos\theta = \dfrac{\vec{s}\cdot\vec{r}}{\|\vec{s}\|\|\vec{r}\|} = \dfrac{-19}{\sqrt{26}\sqrt{30}}$, $\cos\theta < 0$; so θ is obtuse.

$\|\vec{s}\| = \sqrt{1+9+16} = \sqrt{26}$

$\sin\theta = \dfrac{\|\vec{r}\times\vec{s}\|}{\|\vec{r}\|\|\vec{s}\|} = \dfrac{\sqrt{19^2+9+49}}{\sqrt{30}\sqrt{26}}$

$\vec{r}\times\vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 5 \\ -1 & 3 & -4 \end{vmatrix} = (-4-15)\vec{i} - (-8+5)\vec{j} + (6+1)\vec{k} = -19\vec{i} + 3\vec{j} + 7\vec{k}$

c) Find the area of the parallelogram having $\vec{r}$ and $\vec{s}$ as adjacent edges.

$$A = \|\vec{r} \times \vec{s}\| = \sqrt{19^2 + 9 + 49}$$

d) Set $\vec{t} = \vec{i} - \vec{j} + \vec{k}$. Find the volume of the parallelepiped having $\vec{r}$, $\vec{s}$ and $\vec{t}$ as adjacent edges.

$V = |\vec{t} \cdot (\vec{r} \times \vec{s})|$, think about using $\vec{r} \times \vec{s}$ computed in part b).

$= |1(-19) - 1(3) + 1(7)| = |-15| = 15$

e) Describe the surface $x^2 + y^2 + z^2 + 4mx + 2y - 6z + 26 = 0$ according to the values of the parameter m.

Complete the squares: $x^2 + 4mx + 4m^2 + y^2 + 2y + 1 + z^2 - 6z + 9 = -26 + 4m^2 + 1 + 9 = -16 + 4m^2$

Now $-16 + 4m^2 = 4(m^2 - 4) = 4(m-2)(m+2)$

Sign of $m^2 - 4$:

$-\infty$		-2		2		$+\infty$
	$+$	0	$-$	0	$+$	

- If $m < -2$ or $m > 2$, then $m^2 - 4 > 0$, surface is sphere, center $= (-2m, -1, 3)$, radius $= 2\sqrt{m^2 - 4}$

  $(x + 2m)^2 + (y+1)^2 + (z-3)^2 = 4(m^2 - 4)$
- If $-2 < m < 2$, equation represents nothing
- $m = -2$ or $m = 2$, equation represents point $(-2m, -1, 3)$

4. [15] a) Find parametric equations for the line L that is perpendicular to the plane $\mathcal{P} : 2x - 3y + 5z = 6$, and contains the point $A(-1, 2, 4)$.

Since L is perpendicular to $\mathcal{P}$, $\vec{n} = \langle 2, -3, 5 \rangle$ = a normal to $\mathcal{P}$, $\vec{n}$ is parallel to L; so parametric equations for L are

$x = -1 + 2t, \quad y = 2 - 3t, \quad z = 4 + 5t$

b) Let $\vec{u} = -\vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{p} = 2\vec{i} - \vec{j}$, $\vec{q} = \vec{i} + 2\vec{j} - 2\vec{k}$ and $\vec{y} = 2\vec{i} + 4\vec{j} + 5\vec{k}$. Show that the vectors $\vec{p}$, $\vec{q}$ and $\vec{y}$ are pairwise orthogonal, then find three real numbers a, b and c such that $\vec{u} = a\vec{p} + b\vec{q} + c\vec{y}$.

$\vec{p} \cdot \vec{q} = 2(1) - 1(2) = 0$, $\vec{p} \cdot \vec{y} = 2(2) - 1(4) = 0$

$\vec{q} \cdot \vec{y} = 1(2) + 2(4) - 2(5) = 2 + 8 - 10 = 0$; hence $\vec{p}, \vec{q}, \vec{y}$ are pairwise orthogonal.

Now, $\vec{u} \cdot \vec{p} = a\|\vec{p}\|^2 + b\underbrace{\vec{q} \cdot \vec{p}}_{0} + c\underbrace{\vec{y} \cdot \vec{p}}_{0} = a\|\vec{p}\|^2$

Similarly, $\vec{u} \cdot \vec{q} = b\|\vec{q}\|^2$ and $\vec{u} \cdot \vec{y} = c\|\vec{y}\|^2$

Hence

$$a = \frac{\vec{u} \cdot \vec{p}}{\|\vec{p}\|^2} = \frac{-1(2) + 2(-1)}{4 + 1} = -\frac{4}{5}$$

$$b = \frac{\vec{u} \cdot \vec{q}}{\|\vec{q}\|^2} = \frac{-1(1) + 2(2) + 3(-2)}{1 + 4 + 4} = -\frac{3}{9} = -\frac{1}{3}$$

$$c = \frac{\vec{u} \cdot \vec{y}}{\|\vec{y}\|^2} = \frac{-1(2) + 2(4) + 3(5)}{4 + 16 + 25} = \frac{21}{45} = \frac{7}{15}$$

5. [10, **Bonus**] Decide whether the statement is true or false. No explanation needed.

$(\vec{u}\times\vec{v})\cdot\vec{u} = (\vec{u}\times\vec{u})\cdot\vec{v} = \vec{0}\cdot\vec{v} = 0$

$(\vec{u}\times\vec{v})\cdot\vec{v} = (\vec{v}\times\vec{v})\cdot\vec{u} = \vec{0}\cdot\vec{u} = 0$

a) If $\vec{u}$ and $\vec{v}$ are any two vectors, then $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. True

b) If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$. True; $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

c) If $\vec{u}\times\vec{v} = \vec{0}$, then the vectors $\vec{u}$ and $\vec{v}$ are orthogonal. False, $\vec{u} \parallel \vec{v}$

d) If $\vec{u}$ and $\vec{v}$ are any two vectors, then $\|\vec{u}-\vec{v}\| = \|\vec{u}\| - \|\vec{v}\|$. False, pick $\vec{u} = \vec{i}, \vec{v} = \vec{j}$; $\|\vec{u}-\vec{v}\| = \sqrt{2}$, $\|\vec{u}\| - \|\vec{v}\| = 1 - 1 = 0$

e) The graph of $x^2 + 4y^2 = 1$ in 3-space is an ellipse centered at the origin. False, elliptic cylinder

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6. [20] a) Show that the two lines $L_1 : x = 1+t, \quad y = 14+3t, \quad z = -3+2t$, and $L_2 : x = -3-2t, \quad y = 1-4t$, $z = -4-18t$ intersect, and find their point of intersection A.

b) Find the acute angle between the two lines at their point of intersection.

c) Find an equation for the plane determined by the two lines.

a) Let $t = t_1$ in L_1, $t = t_2$ in L_2, and solve for t_1, t_2:

$$1 + t_1 = -3 - 2t_2 \quad (1)$$
$$14 + 3t_1 = 1 - 4t_2 \quad (2)$$
$$-3 + 2t_1 = -4 - 18t_2 \quad (3)$$

Eliminate t_2 from (1) & (2): $-2\cdot(1) + (2)$: $-2 - 2t_1 + 14 + 3t_1 = 6 + 4t_2 + 1 - 4t_2$

Hence $t_1 = 7 - 12 = -5$; $t_1 = -5$ (4)

Report (4) in (1): $1 - 5 = -3 - 2t_2 \to 2t_2 = -3 + 4 = 1 \to t_2 = 1/2$ (5)

Check if (3) is satisfied: LHS $= -3 + 2(-5) = -13 = -4 - 18(\frac{1}{2}) =$ RHS.

So L_1 and L_2 intersect at $A(-4, -1, -13)$.

b) $\vec{u} = \langle 1, 3, 2\rangle \parallel L_1$, $\vec{v} = \langle -2, -4, -18\rangle \parallel L_2$; If θ = acute angle, then

$$\cos\theta = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|\|\vec{v}\|} = \frac{|-2-12-36|}{\sqrt{14}\sqrt{86}\cdot 2} = \frac{25}{\sqrt{14}\sqrt{86}}$$

$$\theta = \cos^{-1}\left(\frac{25}{\sqrt{14}\sqrt{86}}\right)$$

c) $\vec{n} = \vec{u}\times -\frac{1}{2}\vec{v}$ = a normal to plane P determined by L_1 & L_2

$$u\times\left(-\tfrac{1}{2}\vec{v}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 2 \\ 1 & 2 & 9 \end{vmatrix}$$

$$= (27-4)\vec{i} - (9-2)\vec{j} + (2-3)\vec{k}$$

$$= 23\vec{i} - 7\vec{j} - \vec{k}$$

Equation of plane

$$23(x-1) - 7(y-14) - (z+3) = 0$$

You get the point $B(1, 14, -3)$ by setting $t=0$ in L_1.