MAC 2313 (Calculus III) — key
Test 1, Wednesday February 19, 2014

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve any of the points assigned to any question. *You will not get any credit by just writing down the answer to any of the problems without showing the steps.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Total=105pts.**

1. [6] Convert the point $(-\sqrt{3}, -\sqrt{3}, -2)$ from rectangular coordinates to: a) cylindrical coordinates, b) spherical coordinates.

a) $r = \sqrt{3+3} = \sqrt{6}$, $\tan\theta = \frac{-\sqrt{3}}{-\sqrt{3}} = 1$, θ in Quadrant 3, so $\theta = \frac{5\pi}{4}$, $z = -2$

$(\sqrt{6}, \frac{5\pi}{4}, -2)$

b) $\rho = \sqrt{3+3+4} = \sqrt{10}$, $\theta = \frac{5\pi}{4}$ (as in a)), $\phi = \cos^{-1}(-\frac{2}{\sqrt{10}})$

$(\sqrt{10}, \frac{5\pi}{4}, \cos^{-1}(-2/\sqrt{10}))$

2. [13] Let $\vec{w} = -\vec{i} + \vec{j} + 4\vec{k}$, and $\vec{z} = 3\vec{i} + \vec{j} - 2\vec{k}$. a) Find the component of $\vec{z}$ that is parallel to $\vec{w}$. b) If θ is the angle between $\vec{w}$ and $\vec{z}$, find $\sin(\theta)$. c) Find the area of the parallelogram having $\vec{w}$ and $\vec{z}$ as adjacent sides.

a) component of $\vec{z} \parallel \vec{w} = \text{proj}_{\vec{w}}(\vec{z}) = \frac{\vec{z}\cdot\vec{w}}{\|\vec{w}\|^2}\vec{w} = \frac{3(-1)+1(1)-2(4)}{1+1+16}\vec{w} = \frac{-10}{18}\vec{w}$

$= -\frac{5}{9}\vec{w} = \frac{5}{9}\vec{i} - \frac{5}{9}\vec{j} - \frac{20}{9}\vec{k}$

b) $\sin\theta = \frac{\|\vec{w}\times\vec{z}\|}{\|\vec{w}\|\|\vec{z}\|}$

Now $\vec{w}\times\vec{z} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 4 \\ 3 & 1 & -2 \end{vmatrix} = (-2-4)\vec{i} - (2-12)\vec{j} + (-1-3)\vec{k} = -6\vec{i} + 10\vec{j} - 4\vec{k}$

$\sin\theta = \frac{\sqrt{36+100+16}}{\sqrt{18}\sqrt{9+1+4}} = \frac{2\sqrt{38}}{2\sqrt{63}} = \sqrt{38/63}$

c) $A = \|\vec{w}\times\vec{z}\| = \sqrt{36+100+16} = 2\sqrt{38}$

3. [18] Consider the surface $3x^2 - 5y^2 - 2z^2 = 5$. a) Find an equation for the tangent plane to that surface at the point $Q(3, -2, 1)$. b) Find parametric equations of the normal line to that surface at Q. c) Let $g(x, y, z) = xy + y^2z + z^3x$. Find a unit vector in the direction in which g increases most rapidly at the point (2,-1,1). d) Find the directional derivative of g in the direction of the vector $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}$ at the point $(1, 1, 1)$.

a) Set $F(x,y,z) = 3x^2 - 5y^2 - 2z^2 - 5$.

$\nabla F(x,y,z) = 6x\vec{i} - 10y\vec{j} - 4z\vec{k}$

$\nabla F(Q)$ = normal to tangent plane $= 18\vec{i} + 20\vec{j} - 4\vec{k}$

Equation of tangent plane: $18(x-3) + 20(y+2) - 4(z-1) = 0$ or

$9(x-3) + 10(y+2) - 2(z-1) = 0$

b) parametric eqns: $x = 3 + 9t$, $y = -2 + 10t$, $z = 1 - 2t$

c) $\frac{\nabla g(2,-1,1)}{\|\nabla g(2,-1,1)\|}$ = required unit vector. Now $\nabla g(x,y,z) = (y+z^3)\vec{i} + (x+2yz)\vec{j} + (y^2+3z^2x)\vec{k}$

$= \frac{(-1+1)\vec{i} + (2-2)\vec{j} + (1+6)\vec{k}}{7} = \vec{k}$

d) Set $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{2\vec{i} - 3\vec{j} + 4\vec{k}}{\sqrt{4+9+16}}$. $D_{\vec{a}}g(1,1,1) = \nabla g(1,1,1)\cdot\vec{u}$

$= (2\vec{i} + 3\vec{j} + 4\vec{k})\cdot\frac{(2\vec{i} - 3\vec{j} + 4\vec{k})}{\sqrt{29}}$

$= \frac{2(2) + 3(-3) + 4(4)}{\sqrt{29}}$

$= \frac{11}{\sqrt{29}}$

4. [15] Let $f(x,y) = x^3 + y^3 - 6x^2 - 3y^2 + 5$. Find all the critical points of f and classify them as points of local minimum, local maximum, or saddle points.

$f_x(x,y) = 3x^2 - 12x$, $f_y(x,y) = 3y^2 - 6y$. $f_x(x,y) = 0 \to x(3x-12) = 0 \to x = 0$ or $x = 4$

$f_y(x,y) = 0 \to y(3y-6) = 0 \to y = 0$ or $y = 2$. CPs: $(0,0), (0,2), (4,0), (4,2)$

$f_{xx}(x,y) = 6x - 12$, $f_{yy}(x,y) = 6y - 6$, $f_{xy}(x,y) = 0$

$\Delta(x,y) = f_{xy}(x,y)^2 - f_{xx}(x,y) f_{yy}(x,y) = -36(x-2)(y-1)$

C.P	$(0,0)$	$(0,2)$	$(4,0)$	$(4,2)$
$\Delta(x_0,y_0)$	-72	72	72	-72
$f_{xx}(x_0,y_0)$	-12			12
Classification	local maximum	saddle point	saddle point	local minimum

5. [15] a) Write down the definition of " f is differentiable at (x_0, y_0)".

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - h f_x(x_0,y_0) - k f_y(x_0,y_0)}{\sqrt{h^2+k^2}} = 0$$

b) Use implicit differentiation to find the partial derivatives $\partial x/\partial y$ and $\partial x/\partial z$ if $x^2 + z^3 + \cos(xy) = 1$.

$\frac{\partial}{\partial y}(x^2 + z^3 + \cos(xy)) = \frac{\partial}{\partial y}(1) = 0$

$2x\frac{\partial x}{\partial y} + 0 + (y\frac{\partial x}{\partial y} + x)(-\sin(xy)) = 0$; $(2x - y\sin(xy))\frac{\partial x}{\partial y} = x\sin(xy)$

$\frac{\partial x}{\partial y} = x\sin(xy)/(2x - y\sin(xy))$

$\frac{\partial}{\partial z}(x^2 + z^3 + \cos(xy)) = \frac{\partial}{\partial z}(1) = 0 \to 2x\frac{\partial x}{\partial z} + 3z^2 - y\frac{\partial x}{\partial z}\sin(xy) = 0$

$$\frac{\partial x}{\partial z} = \frac{-3z^2}{2x - y\sin(xy)}$$

6. [10] If $f = f(x,y)$ is a differentiable function, $x = uv$, $y = u^2 - v^2$, and $w(u,v) = f(uv, u^2 - v^2)$, use the chain rule to find the partial derivatives w_u and w_v.

$w_u = f_x(uv, u^2-v^2)x_u + f_y(uv, u^2-v^2)y_u$, $x_u = v$, $y_u = 2u$

$= f_x(uv, u^2-v^2)v + 2f_y(uv, u^2-v^2)u$ $\quad x_v = u$, $y_v = -2v$

$w_v = f_x(uv, u^2-v^2)x_v + f_y(uv, u^2-v^2)y_v$

$= f_x(uv, u^2-v^2)u - 2f_y(uv, u^2-v^2)v$

7. [12] Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} - 2x + 3y, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

Find $f_x(0,0)$, and $f_y(0,0)$. Is f differentiable at $(0,0)$?

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 - 2h + 0 - 0}{h} = -2$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0 - 0 + 3k - 0}{k} = 3$$

Method 1: $$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - h f_x(0,0) - k f_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\frac{hk}{h^2+k^2} - 2h + 3k + 2h - 3k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{hk}{(h^2+k^2)^{3/2}} \qquad h = r\cos\theta,\ k = r\sin\theta,\ r>0,\ 0\le\theta<2\pi$$

$$= \lim_{r\to 0^+} \frac{r^2\cos\theta\sin\theta}{r^3}$$

$$= \lim_{r\to 0^+} \frac{1}{r}\cos\theta\sin\theta$$

$$= +\infty \text{ for } \theta = \frac{\pi}{4}$$

So f is not differentiable at $(0,0)$.

Method 2: $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} f(x,y) = \lim_{y\to 0} 3y = 0$, $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=y}} f(x,y) = \lim_{y\to 0} \frac{y^2}{2y^2} + y = \frac{1}{2}$

So $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE; hence f is not continuous at $(0,0)$; so f is not differentiable at $(0,0)$.

8. [16] a) Find an equation for the plane that contains the point $(1,1,1)$ and the line $L: x = 1+2t,\quad y = 2-3t,\quad z = -2+4t$.

b) Show that the two lines $L_1: x = -1+t,\quad y = 2-2t,\quad z = -4+3t$ and $L_2: x = 7+2t,\quad y = 1+t,\quad z = -1-t$ intersect and find their point of intersection.

a) $\vec{u} = \langle 2, -3, 4 \rangle$ is parallel to L

$B(1,2,-2)$ lies on L (set $t=0$ to get B)

If we set $A(1,1,1)$. Then $\vec{AB} = \langle 1-1, 2-1, -2-1 \rangle = \langle 0, 1, -3 \rangle$

$\vec{n} = \vec{u} \times \vec{AB}$ = a normal to plane. $\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 0 & 1 & -3 \end{vmatrix} = (9-4)\vec{i} - (-6-0)\vec{j} + (2-0)\vec{k} = 5\vec{i} + 6\vec{j} + 2\vec{k}$

Equation of plane: $5(x-1) + 6(y-1) + 2(z-1) = 0$

b) Do there exist t_1 and t_2 such that

(1) $-1 + t_1 = 7 + 2t_2$

(2) $2 - 2t_1 = 1 + t_2$

(3) $-4 + 3t_1 = -1 - t_2$?

$2\cdot(1) + (2)$: $\underbrace{-2 + 2t_1 + 2 - 2t_1}_{=0} = 14 + 4t_2 + 1 + t_2 \to 5t_2 + 15 = 0 \to t_2 = -3$ (4)

(4) in (1): $t_1 = 8 + 2t_2 = 8 + 2(-3) = 8 - 6 = 2$ (5)

(5) in LHS of (3): $-4 + 3(2) = 2$

(4) in RHS of (3): $-1 - (-3) = 2$

so LHS of (3) = RHS of (3); hence L_1 & L_2 intersect

To find the point of intersection plug $t_1 = 2$ into equations of L_1 or $t_2 = -3$ into equations of L_2

You get in either case the point $(1, -2, 2)$.