MAC 2313 (Calculus III) — Answers

Test 1, February 4, 2015

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good Luck.

1 [5] Describe the given surface; if it is a sphere, state its radius and center. If it is a point, state its coordinates. $x^2+y^2+z^2-4x+8y-2z-4=0$.

$x^2-4x+4+y^2+8y+16+z^2-2z+1=4+4+16+1=25$

$(x-2)^2+(y+4)^2+(z-1)^2=25$; surface is the sphere centered at $(2,-4,1)$ with radius $r=5$.

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2. [10] Set $\vec{u}=\vec{i}+\vec{j}+3\vec{k}$, $\vec{v}=-2\vec{i}-\vec{j}+\vec{k}$ and $\vec{w}=-3\vec{i}-\vec{j}+5\vec{k}$. Show that $\vec{u}$ and $\vec{v}$ are orthogonal vectors, and find two scalars a and b such that $\vec{w}=a\vec{u}+b\vec{v}$.

$\vec{u}\cdot\vec{v}=1(-2)+1(-1)+3(1)=-3+3=0$; so $\vec{u}\perp\vec{v}$,

$\vec{w}\cdot\vec{u}=a\|\vec{u}\|^2+b\vec{v}\cdot\vec{u}=a\|\vec{u}\|^2 \to a=\frac{\vec{w}\cdot\vec{u}}{\|\vec{u}\|^2}=\frac{1(-3)+1(-1)+3(5)}{1+1+9}=\frac{11}{11}=1$

$\vec{w}\cdot\vec{v}=a\vec{u}\cdot\vec{v}+b\|\vec{v}\|^2=b\|\vec{v}\|^2 \to b=\frac{\vec{w}\cdot\vec{v}}{\|\vec{v}\|^2}=\frac{-3(-2)-1(-1)+5(1)}{4+1+1}=\frac{12}{6}=2$

Hence $\vec{w}=\vec{u}+2\vec{v}$

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3. [12] Let $\vec{r}=2\vec{i}-3\vec{j}+4\vec{k}$, and $\vec{u}=-2\vec{i}+\vec{j}-4\vec{k}$. a) Find the vector component of $\vec{u}$ that is parallel to $\vec{r}$.

component of $\vec{u}$ parallel to $\vec{r}=\text{Proj}_{\vec{r}}(\vec{u})=\frac{\vec{u}\cdot\vec{r}}{\|\vec{r}\|^2}\vec{r}=\frac{-2(2)+1(-3)-4(4)}{4+9+16}\vec{r}$

$=-\frac{23}{29}\vec{r}$

b) If θ is the angle between $\vec{r}$ and $\vec{u}$, find $\cos(\theta)$ and $\sin(\theta)$.

$\cos\theta=\frac{\vec{r}\cdot\vec{u}}{\|\vec{r}\|\|\vec{u}\|}=\frac{2(-2)-3(1)+4(-4)}{\sqrt{29}\sqrt{4+1+16}}=\frac{-23}{\sqrt{29}\sqrt{21}}$; $\sin\theta=\frac{\|\vec{u}\times\vec{r}\|}{\|\vec{u}\|\|\vec{r}\|}=\frac{\sqrt{64+16}}{\sqrt{29}\sqrt{21}}=\frac{\sqrt{80}}{\sqrt{29}\sqrt{21}}$

$\vec{u}\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 1 & -4\\ 2 & -3 & 4\end{vmatrix}=(4-12)\vec{i}-(-8+8)\vec{j}+(6-2)\vec{k}$

$=-8\vec{i}+4\vec{k}$

c) If the force $\vec{F}=\vec{r}$ is applied to move an object 8 meters in the direction of the vector $\vec{u}$, find the work done by $\vec{F}$.

Work $=-\|\text{Proj}_{\vec{u}}(\vec{r})\|\,d$, $d=8m$,

$=\frac{(\vec{u}\cdot\vec{r})}{\|\vec{u}\|^2}\|\vec{u}\|\,d=\frac{\vec{u}\cdot\vec{r}}{\|\vec{u}\|}d=-\frac{23(8)}{\sqrt{21}}=-\frac{184}{\sqrt{21}}$ J

as $\text{Proj}_{\vec{u}}(\vec{r})$ is oppositely directed to the direction of motion,

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4. [8] Set $\vec{u}=\vec{i}-3\vec{k}$, $\vec{v}=-\vec{j}+\vec{k}$ and $\vec{w}=2\vec{i}-\vec{j}$. a) Find the area of the triangle having $\vec{u}$ and $\vec{w}$ as adjacent sides. b) Find the volume of the parallelepiped having $\vec{u}$, $\vec{v}$ and $\vec{w}$ as adjacent edges.

a) $A=\frac{\|\vec{u}\times\vec{w}\|}{2}$. $\vec{u}\times\vec{w}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -3\\ 2 & -1 & 0\end{vmatrix}=(0-3)\vec{i}-(0+6)\vec{j}+(-1-0)\vec{k}$

$=-3\vec{i}-6\vec{j}-\vec{k}$

$=\frac{\sqrt{9+36+1}}{2}$

$=\frac{\sqrt{46}}{2}$

b) $V=|\vec{v}\cdot(\vec{u}\times\vec{w})|$

$=|-1(-6)+1(-1)|=5$

5. [20] a) Show that the two lines $L_1 : x = 1 + 2t, \quad y = -2 + 2t, \quad z = 3 + t$, and $L_2 : x = 2 + 15t, \quad y = -4 + 12t$, $z = 1 + 5t$ intersect, and find their point of intersection A.
b) Find an equation for the plane $\mathcal{P}$ that contains both L_1 and L_2.
c) Find the distance between the plane $\mathcal{P}$ and the point $B(1, 2, -1)$.

a) Find t_1 and t_2 with

$1 + 2t_1 = 2 + 15t_2 \quad (1)$

$-2 + 2t_1 = -4 + 12t_2 \quad (2)$

$3 + t_1 = 1 + 5t_2 \quad (3)$

Subtract (2) from (1) side by side:

$1 - (-2) + 2t_1 - 2t_1 = 2 + 4 + 15t_2 - 12t_2$

$3 = 6 + 3t_2 \rightarrow t_2 = -1 \quad (4)$

Report (4) in (2):

$-2 + 2t_1 = -4 - 12 = -16$

$2t_1 = -14 \rightarrow t_1 = -7 \quad (5)$

RHS of (3) $= 1 + 5(-1) = -4$

LHS of (3) $= 3 + (-7) = -4$

RHS = LHS; so L_1 & L_2 intersect at $A = (-13, -16, -4)$

b) $\vec{u_1} = \langle 2, 2, 1 \rangle \parallel L_1$, $\vec{u_2} = \langle 15, 12, 5 \rangle \parallel L_2$

$\vec{n} = \vec{u_1} \times \vec{u_2}$ = a normal to $\mathcal{P}$.

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ 15 & 12 & 5 \end{vmatrix} = (10 - 12)\vec{i} - (10 - 15)\vec{j} + (24 - 30)\vec{k}$$

$$= -2\vec{i} + 5\vec{j} - 6\vec{k}$$

$B = (1, -2, 3)$ lies on plane; equ of $\mathcal{P}$: $-2(x - 1) + 5(y + 2) - 6(z - 3) = 0$

or $2x - 5y + 6z - 30 = 0$

c) $$d(B, \mathcal{P}) = \frac{|2(1) - 5(2) + 6(-1) - 30|}{\sqrt{4 + 25 + 36}} = \frac{44}{\sqrt{65}}$$

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6. [4] Find an equation for the surface that results when the hyperbolic paraboloid $z = 3x^2 - 8y^2$ is reflected about the plane: i) $z = 0$, ii) $y = z$. Identify the surface obtained in either case.

i) $-z = 3x^2 - 8y^2$ or $z = 8y^2 - 3x^2$; hyperbolic paraboloid along y-axis

ii) $y = 3x^2 - 8z^2$; hyperbolic paraboloid along x-axis.

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7 [6]. a) Convert from rectangular to cylindrical coordinates: i) $(-\sqrt{3}, 3, -4)$, b) Convert the equation $\rho \sin\phi = 2$ from spherical to rectangular coordinates, and identify the surface.

a) $r = \sqrt{3 + 9} = \sqrt{12} = 2\sqrt{3}$

$\tan\theta = \frac{3}{-\sqrt{3}} = -\sqrt{3}$

θ in Q II; so $\theta = \frac{2\pi}{3}$

$z = -4$

$(2\sqrt{3}, \frac{2\pi}{3}, -4)$

b) $r = 2$

$r^2 = 4$

$x^2 + y^2 = 4$ circular cylinder along z-axis