MAC 2313 (Calculus III) — Answers
Test 1, Friday May 19, 2017

Name: **PID:**

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work. Guessing the correct answers won't give you any credits. You must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may use the back of the page to show your work. Total=60 points.

1. [10] a) Find an equation for the plane $\mathcal{P}$ that passes through the point $A(1,-2,-3)$ and is perpendicular to the line $L: x = 2-t,\quad y = 4+3t,\quad z = 3+2t$.

$\vec{u} = \langle -1, 3, 2\rangle$ = a normal vector to the plane, since L is perpendicular to the plane $\mathcal{P}$.
Equation: $-(x-1)+3(y+2)+2(z+3)=0$.

b) Find the intersection of the plane $Q: 2x-3y+4z=5$ and the line L.

Solve for t: $2(2-t)-3(4+3t)+4(3+2t)=5$, to find the intersection.

$$4-2t-\cancel{12}-9t+\cancel{12}+8t=5$$
$$-3t = 5-4 = 1 \rightarrow t = -1/3$$

Point of intersection $= (2+\frac{1}{3}, 4-1, 3-\frac{2}{3}) = (\frac{7}{3}, 3, \frac{7}{3})$

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2. [10] a) If a point A is given in spherical coordinates as (ρ,θ,ϕ), write down the rectangular coordinates (x,y,z) of A.

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

b) Convert the equation $\rho^2\cos(2\phi)=1$ from spherical to rectangular coordinates, and identify the surface.

$\cos(2\phi) = \cos^2\phi - \sin^2\phi$
$\rho^2\cos^2\phi - \rho^2\sin^2\phi = 1$, $z^2-(x^2+y^2)=1$ or $z^2-x^2-y^2=1$;
hyperboloid of two sheets

c) Convert the point $(-5,-5,3)$ from rectangular to cylindrical coordinates.

$r = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$; $\tan\theta = \frac{-5}{-5} = 1$, θ in QIII; so $\theta = \frac{5\pi}{4}$; $z=3$

$(5\sqrt{2}, \frac{5\pi}{4}, 3)$

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3. [20] Let $\vec{r} = -\vec{i}+\vec{j}-2\vec{k}$, and $\vec{s} = 3\vec{i}-2\vec{j}+\vec{k}$. a) Find the orthogonal projection of $\vec{r}$ onto $\vec{s}$.

$$\text{Proj}_{\vec{s}}(\vec{r}) = \frac{\vec{r}\cdot\vec{s}}{\|\vec{s}\|^2}\vec{s} = \frac{-3-2-2}{9+4+1}\vec{s} = -\frac{1}{2}\vec{s} = -\frac{3}{2}\vec{i}+\vec{j}-\frac{1}{2}\vec{k}$$

b) If θ is the angle between $\vec{r}$ and $\vec{s}$, find $\cos(\theta)$ and $\sin(\theta)$. Is θ acute or obtuse? $\rightarrow$

$$\vec{s}\times\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ -1 & 1 & -2 \end{vmatrix} = (4-1)\vec{i} - (-6+1)\vec{j} + (3-2)\vec{k} = 3\vec{i}+5\vec{j}+\vec{k}$$

$$\cos\theta = \frac{\vec{r}\cdot\vec{s}}{\|\vec{r}\|\|\vec{s}\|} = \frac{-7}{\sqrt{6}\sqrt{14}}$$

θ is obtuse, since $\cos\theta < 0$

$$\sin\theta = \frac{\|\vec{s}\times\vec{r}\|}{\|\vec{s}\|\|\vec{r}\|} = \frac{\sqrt{9+25+1}}{\sqrt{14}\sqrt{6}} = \frac{\sqrt{35}}{\sqrt{84}}$$

c) Find a vector $\vec{u}$ with norm 5 that is oppositely directed to $\vec{r}$.

$$\vec{u} = 5\left(-\frac{\vec{r}}{\|\vec{r}\|}\right) = -\frac{5}{\sqrt{6}}\left(-\vec{i}+\vec{j}-2\vec{k}\right) = \frac{5}{\sqrt{6}}\vec{i} - \frac{5}{\sqrt{6}}\vec{j} + \frac{10}{\sqrt{6}}\vec{k}$$

d) Set $\vec{t} = \vec{i} + \vec{j} + \vec{k}$. Find the volume of the parallelepiped having $\vec{r}$, $\vec{s}$ and $\vec{t}$ as adjacent edges.

$V = |\vec{t}\cdot(\vec{r}\times\vec{s})| = |\vec{t}\cdot(\vec{s}\times\vec{r})|$, remember to use part b) here.

$\vec{t}\cdot(\vec{s}\times\vec{r}) = \langle 1,1,1\rangle\cdot\langle 3,5,1\rangle = 3+5+1 = 9$

$V = 9$

e) Find an equation for the sphere containing the point $B(1,-2,3)$ and centered at $D(-1,1,2)$.

r = radius of sphere $= d(B,D) = \sqrt{(-1-1)^2+(1+2)^2+(2-3)^2} = \sqrt{4+9+1} = \sqrt{14}$

Eqn: $(x+1)^2 + (y-1)^2 + (z-2)^2 = 14$

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4. [10] Show that the two lines $L_1: x = 1+t, \quad y = -3+2t, \quad z = 14+3t$, and $L_2: x = -4-t, \quad y = 5+4t, \quad z = 23+5t$ intersect, and find their point of intersection A.

• Show L_1 and L_2 intersect: find t_1 and t_2 with

$1 + t_1 = -4 - t_2 \quad (1)$

$-3 + 2t_1 = 5 + 4t_2 \quad (2)$

$14 + 3t_1 = 23 + 5t_2 \quad (3)$

$4\cdot(1) + (2)$ yields

$4 - 3 + 6t_1 = -16 + 5$

$6t_1 = -12 \to t_1 = -2 \quad (4)$

(4) in (1) yields

$1 - 2 = -4 - t_2$; so $t_2 = -3 \quad (5)$

We have used (1) & (2) to get t_1 and t_2. We now use (3) to show that L_1 & L_2 intersect

$LHS = 14 + 3(-2) = 8 = 23 + 5(-3) = RHS$; So L_1 and L_2 intersect

$A(-1,-7,8)$, by using t_1 in L_1 or t_2 in L_2.

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5. [10, **Bonus**] Decide whether the statement is true or false. No explanation needed.

a) The graph of $x^2 + y^2 = 16$ in 3-space is the circle centered at the origin with radius 4. False; cylinder along z-axis

b) If $\vec{u}$ and $\vec{v}$ are any two vectors, then $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$. False (See Triangle inequality)

c) If a point belongs to both the xy-plane and the yz-plane, then the point lies on the y-axis. True, $(x,y,0) = (0,y,z)$; so $x=0, z=0$

d) If $\vec{u}$ is a unit vector that is parallel to a nonzero vector $\vec{v}$, then $\vec{u}\cdot\vec{v} = \pm\|\vec{v}\|$. True, $\vec{u} = \pm\frac{\vec{v}}{\|\vec{v}\|}$

$\vec{u}\cdot\vec{v} = \pm\frac{\|\vec{v}\|^2}{\|\vec{v}\|} = \pm\|\vec{v}\|$

e) If $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w}$ and $\vec{u} \neq \vec{0}$, then $\vec{v} = \vec{w}$. False, just pick $\vec{u} = \vec{i}$, $\vec{v} = \vec{j}$, $\vec{w} = \vec{k}$

$\vec{u}\cdot\vec{v} = 0 = \vec{u}\cdot\vec{w}$ but $\vec{v} \neq \vec{w}$.