MAP 2302 (Differential Equations)
TEST 1, Thursday February 11, 2010 — Key

Name: PID:

**Remember that no documents or calculators are allowed during the test. You shall show all your work to deserve the full mark assigned to any question. 5 pages. Total=100 points**

1. [11+10] a) Show that the function given by $f(x) = 2e^{-2x} + \sin x - \cos x$ is the solution of the initial-value problem: $y'' + 4y' + 5y = 2e^{-2x} + 8\sin x$, $y(0) = 1$, $y'(0) = -3$. b) Show that the differential equation: $(x^2y + 2y - 3)dx + xdy = 0$ is not exact. b1) Find an integrating factor for that equation. b2) Write down the exact differential equation, but do not solve it.

a) $f'(x) = -4e^{-2x} + \cos x + \sin x$

$f''(x) = 8e^{-2x} - \sin x + \cos x$

$$f''(x) + 4f'(x) + 5f(x) = 8e^{-2x} - \sin x + \cancel{\cos x} - 16e^{-2x} + 4\cancel{\cos x} + 4\sin x + 10e^{-2x} + 5\sin x - 5\cancel{\cos x}$$

$$= 2e^{-2x} + 8\sin x$$

So $f$ solves the D.E.

$f(0) = 2e^0 + \sin 0 - \cos 0 = 2 - 1 = 1$

$f'(0) = -4e^0 + \cos 0 + \sin 0 = -4 + 1 = -3$

Hence $f$ solves the IVP.

b) Set $M(x,y) = x^2y + 2y - 3$, $N(x,y) = x$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x^2 + 2 - 1 = x^2 + 1$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{x^2+1}{x} = \frac{1}{x} + x$$

Integrating factor: $\mu(x) = e^{\int \frac{1}{x} + x \, dx}$

$= e^{(\ln|x|) + \frac{x^2}{2}}$

$= |x| e^{\frac{x^2}{2}}$

Disregarding the absolute value, we write $\mu(x) = xe^{\frac{x^2}{2}}$,

b2) The exact D.E.:

$$\underbrace{xe^{\frac{x^2}{2}}(x^2y + 2y - 3)}_{\hat{M}(x,y)}dx + \underbrace{x^2e^{\frac{x^2}{2}}}_{\hat{N}(x,y)}dy = 0$$

$$\frac{\partial \hat{M}}{\partial y}(x,y) = (x^3 + 2x)e^{\frac{x^2}{2}} = \frac{\partial \hat{N}}{\partial x}(x,y).$$

2. [10] State Theorem 1.1 from the text. Use that theorem to show that the initial-value problem:

$$\begin{cases} \frac{dy}{dx} = 2^{xy} - y^3x^2 \\ y(4) = \pi. \end{cases}$$

has a unique solution defined on some sufficiently small interval $|x-4| \le h$ about $x_0 = 4$.

Set $f(x,y) = 2^{xy} - y^3x^2$. Then $\frac{\partial f}{\partial y}(x,y) = x(\ln 2)2^{xy} - 3y^2x^2$.

i) $f$ and $\frac{\partial f}{\partial y}$ are continuous everywhere in the $xy$-plane. So $f$ and $\frac{\partial f}{\partial y}$ are continuous on every domain $D$ that contains the point $(4, \pi)$.

ii) Theorem 1.1 then shows that the given IVP has a unique solution defined on some sufficiently small interval $|x-4| \le h$ about $x_0 = 4$.

Note: For the Statement of Theorem 1.1, see textbook or notes.

3. [12] Solve the initial-value problem: $(x^2+1)\frac{dy}{dx} + 2xy = x^3, \quad y(0) = 2$.

D.E may be written:

$$\frac{dy}{dx} + \frac{2x}{x^2+1}y = \frac{x^3}{x^2+1} \quad (\text{linear D.E})$$

Set $P(x) = \frac{2x}{x^2+1}$, $Q(x) = \frac{x^3}{x^2+1}$

$$y = \left[\int Q(x) e^{\int P(x)dx} dx + c\right] e^{-\int P(x)dx}, \quad c = \text{constant}$$

$$= \left[\int \frac{x^3}{x^2+1} e^{\int \frac{2x\,dx}{x^2+1}} dx + c\right] e^{-\int \frac{2x}{x^2+1}dx}$$

$$= \left[\int \frac{x^3}{x^2+1} e^{\ln(x^2+1)} dx + c\right] e^{-\ln(x^2+1)}$$

$$= \left[\int x^3 \left(\frac{x^2+1}{x^2+1}\right) dx + c\right] e^{\ln\left(\frac{1}{x^2+1}\right)}$$

$$= \left[\int x^3 dx + c\right] \frac{1}{x^2+1}$$

$$= \left[\frac{x^4}{4} + c\right] \frac{1}{x^2+1}.$$

Now:

$$y(0) = c = 2$$

So, soln of IVP:

$$y = \left(\frac{x^4}{4} + 2\right)\frac{1}{x^2+1}$$

4. [15] Given that $y = x^2$ solves the differential equation: $x^2y'' - 6xy' + 10y = 0$, use the method of reduction of order to find a linearly independent solution. Write down the general solution.

Seek a linearly independent solution

$z = x^2 v$. So

$z' = 2x v + x^2 v'$, $z'' = 2v + 2xv' + 2xv' + x^2 v''$

$$x^2 z'' - 6x z' + 10 z = \cancel{2x^2 v} + 4x^3 v' + x^4 v'' - \cancel{12x^2 v} - 6x^3 v' + \cancel{10x^2 v}$$
$$= x^4 v'' - 2x^3 v'$$
$$= 0$$

Set $u = v'$. Then

$x^4 u' - 2x^3 u = 0$ or $x u' - 2u = 0$

or $u' - \frac{2}{x} u = 0$ (linear DE, $P = -\frac{2}{x}$, $Q = 0$)

$$u = c e^{-\int P(x)dx} = c e^{\int \frac{2}{x} dx} = c e^{2\ln|x|} = c e^{\ln(x^2)} = cx^2$$

$v' = cx^2$

$v = \frac{c}{3} x^3 + d$; Choose $d = 0$, $c = 3$ to get $z = x^5$

Hence $z = x^5$ is a linearly indept soln of the given DE

general solution $y = C_1 x^2 + C_2 x^5$, $C_1, C_2$ are arbitrary constants.

5. [12+10] a) Solve the homogeneous differential equation: $(x^2+3y^2)dx - 2xydy = 0$.

b) Reduce the equation $(x+3y-7)dx + (4x+12y+8)dy = 0$ to a separable equation. Do not solve the separable equation obtained.

a) Set $y = xv$. Then $dy = xdv + vdx$. D.E. becomes:

$$(x^2+3x^2v^2)dx - 2x^2v(xdv+vdx) = 0$$

$$(x^2+3x^2v^2-2x^2v^2)dx - 2x^3v\,dv = 0$$

$$(x^2+x^2v^2)dx - 2x^3v\,dv = 0$$

$$x^2(1+v^2)dx - 2x^3v\,dv = 0$$

Divide by $x^3(1+v^2)$, $(x \neq 0)$:

$$\frac{dx}{x} - \frac{2v}{1+v^2}dv = 0$$

Integrating:

$$\int\frac{dx}{x} - \int\frac{2v}{1+v^2}dv = c, \quad c = \text{constant}$$

$$\ln|x| - \ln(1+v^2) = \ln k, \quad k>0$$

$$\ln\frac{|x|}{1+v^2} = \ln k \rightarrow \frac{|x|}{1+v^2} = k, \quad k \geq 0$$

Soln: $\dfrac{|x|}{1+\frac{y^2}{x^2}} = k$

b) Set $z = x+3y$. Then $dz = dx + 3dy$. D.E. becomes:

$$(z-7)dx + (4z+8)\left(\frac{dz-dx}{3}\right) = 0$$

$$(3z-21)dx - (4z+8)dx + (4z+8)dz = 0$$

$(-z-29)dx + (4z+8)dz = 0$, which is a separable DE in $x$ and $z$

or

$$(z-7)(dz-3dy) + (4z+8)dy = 0$$

$$(z-7)dz + (4z+8-3z+21)dy = 0$$

$(z-7)dz + (z+29)dy = 0$, separable in $y$ and $z$.

6. [10] Find the orthogonal trajectories to the family of curves $x^2 - 4y^2 = c$.

Differentiate implicitly w.r.t. $x$:

$$\frac{d}{dx}(x^2-4y^2)=\frac{d}{dx}(c)=0$$

$$2x-8y\frac{dy}{dx}=0 \to \frac{dy}{dx}=\frac{x}{4y}$$

D.E. for orthogonal trajectories:

$$\frac{dy}{dx}=-\frac{4y}{x}$$

$\frac{1}{y}\frac{dy}{dx}=-\frac{4}{x}$ or $\frac{1}{y}dy=-\frac{4}{x}dx$. Integrating:

$$\int\frac{1}{y}dy=-\int\frac{4}{x}dx$$

$\ln|y|=-4\ln|x|+c_1$, $c_1$ = arbitrary constant

$\ln(|y|x^4)=c_1=\ln k$, $k>0 \to |y|x^4=k$, $k>0$

$\Rightarrow yx^4=\ell$, $\ell$ arbitrary constant.

7. [10] Find the constant $A$ such that the differential equation: $(3x^2y^2 + Ay)dx + (2x^3y + 4x - \sin y)dy = 0$ is exact. Solve the exact differential equation.

$$\frac{\partial}{\partial y}(3x^2y^2+Ay)=\frac{\partial}{\partial x}(2x^3y+4x-\sin y)$$

$6x^2y+A=6x^2y+4$; hence $A=4$. D.E is exact if and only if $A=4$.

We now seek a function $F=F(x,y)$ such that

(i) $\frac{\partial F}{\partial x}(x,y)=3x^2y^2+4y$, and (ii) $2x^3y+4x-\sin y=\frac{\partial F}{\partial y}(x,y)$

Integrating (i) in $x$:

$$F(x,y)=\int(3x^2y^2+4y)dx$$

$$=x^3y^2+4xy+C(y) \quad \text{(iii)}$$

Differentiate (iii) w.r.t. $y$:

$$\frac{\partial F}{\partial y}(x,y)=2x^3y+4x+C'(y)$$

$$=2x^3y+4x-\sin y, \text{ by (ii)}$$

Hence $C'(y)=-\sin y$; so $C(y)=\cos y$

Soln of D.E.: $x^3y^2+4xy+\cos y=c$, $c$ = arbitrary constant.