MAP 2302 (Differential Equations) – key
TEST 1, Friday February 21, 2014

Name: PID:

**Remember that no documents or calculators are allowed during the test. You must show all your work to deserve the full credit assigned to any question. 4 pages.**

1.[5] Given that 2,2,-1,-1,-1,4,2-5i,2+5i,2-5i,2+5i are the roots of the auxiliary equation corresponding to some 10th-order homogeneous linear differential equation with constant coefficients, write down the general solution of the differential equation.

$$y = (c_1+c_2x)e^{2x} + (c_3+c_4x+c_5x^2)e^{-x} + c_6e^{4x} + (c_7+c_8x)\cos(5x)e^{2x}$$
$$+ (c_9+c_{10}x)\sin(5x)e^{2x}$$

$c_1, c_2, \dots, c_{10}$ = constants

2. [20] Solve the initial-value problem: $(3x^2y + e^{2x})dx + (x^3 + \cos y)dy = 0, \quad y(0) = \pi/2$.

$M(x,y) = 3x^2y + e^{2x}$, $N(x,y) = x^3 + \cos y$

$\frac{\partial M}{\partial y}(x,y) = 3x^2 = \frac{\partial N}{\partial x}(x,y)$; D.E. is exact.

By regrouping: $\underbrace{3x^2y\,dx + x^3dy}_{d(x^3y)} + \underbrace{e^{2x}dx}_{d(\frac{e^{2x}}{2})} + \underbrace{\cos y\,dy}_{d(\sin y)} = 0$

$d(x^3y) + d(\frac{e^{2x}}{2}) + d(\sin y) = d(c)$, $c$ = constant

$d(x^3y + \frac{e^{2x}}{2} + \sin y) = d(c)$

General solution: $x^3y + \frac{e^{2x}}{2} + \sin y = c$

standard method: There exists a function $F$ with $dF = M\,dx + N\,dy$; so

(i) $F_x(x,y) = 3x^2y + e^{2x}$, (ii) $F_y = x^3 + \cos y$

Integrating (i) w.r.t. $x$:

$F(x,y) = \int 3x^2y + e^{2x}dx = x^3y + \frac{e^{2x}}{2} + k(y)$ (iii)

Differentiate (iii) w.r.t. $y$ and use (ii):

$F_y(x,y) = x^3 + k'(y)$
$= x^3 + \cos y$, by (ii); So $k'(y) = \cos y$; hence $k(y) = \int \cos y\,dy = \sin y$

General solution: $x^3y + \frac{e^{2x}}{2} + \sin y = c$, $c$ = constant

When $x=0$, $y = \pi/2$: $0(\frac{\pi}{2}) + \frac{e^0}{2} + \sin\frac{\pi}{2} = c \rightarrow \frac{1}{2} + 1 = c$

Solution of IVP: $x^3y + \frac{e^{2x}}{2} + \sin y = \frac{3}{2}$.

3. [15] Given that $y = \cos x$ solves the differential equation: $y^{(4)} + 3y''' + 3y'' + 3y' + 2y = 0$, find the general solution of the differential equation.

If $\cos x$ solves the D.E, then $\pm i$ is a solution of the auxiliary eqn:

$m^4 + 3m^3 + 3m^2 + 3m + 2 = 0$

Consequently $m^4 + 3m^3 + 3m^2 + 3m + 2 = (m^2+1)(m^2+3m+2)$, since $(m-i)(m+i) = m^2+1$

$= (m^2+1)(m+1)(m+2)$

The roots of the auxiliary equations are: $m_1 = i, m_2 = -i, m_3 = -1, m_4 = -2$

General solution of D.E:

$y = c_1 \cos x + c_2 \sin x + c_3 e^{-x} + c_4 e^{-2x}$, $c_1, c_2, c_3, c_4$ = Constants

4. [20] a) Reduce the equation $(x^2+1)\frac{dy}{dx} - 2xy = y^{-3}\sec^2 x$ to a linear equation. Do not solve the linear equation obtained. b) Reduce the equation $(x - 3y + 5)dx + (3x - 9y + 11)dy = 0$ to a separable equation. Do not solve the separable equation obtained.

a) D.E may be recast as: $(x^2+1)y^3\frac{dy}{dx} - 2xy^4 = \sec^2 x$.

Set $z = y^4$. Then $\frac{dz}{dx} = 4y^3\frac{dy}{dx}$; the linear D.E. is

$$\frac{(x^2+1)}{4}\frac{dz}{dx} - 2xz = \sec^2 x$$

$a_1 = 1, b_1 = -3, a_2 = 3, b_2 = -9; a_1 b_2 = -9 = b_1 a_2$.

b) Set $z = x - 3y$. Then $dz = dx - 3dy$, and $dy = \frac{dx - dz}{3}$

D.E. becomes

$$(z+5)dx + (3z+11)\left(\frac{dx-dz}{3}\right) = 0$$

$$(3(z+5) + 3z + 11)dx + (-1)(3z+11)dz = 0$$

$(6z+26)dx - (3z+11)dz = 0$, which is separable.

5. [20] Use the method of undetermined coefficients to solve the differential equation: $y'' - 2y' + y = 22e^x$

Homogeneous equation: $y'' - 2y' + y = 0$

Auxiliary equation: $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m_1 = 1,\ m_2 = 1$$

$y_c = c_1 e^x + c_2 x e^x$, $c_1, c_2$ = constants

Only one UC set $S_{e^x} = \{e^x\}$. Now $e^x$ and $xe^x$ both solve the homogeneous D.E. So $S_{e^x_{new}} = \{x^2 e^x\}$. Seek $y_p = Ax^2e^x$, $A$ = constant to be determined

$$y_p' = 2Axe^x + Ax^2e^x$$

$$y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$$

$$\begin{aligned} y_p'' - 2y_p' + y_p &= (Ax^2 + 4Ax + 2A)e^x - 2(Ax^2 + 2Ax)e^x + Ax^2e^x \\ &= (\cancel{Ax^2} - \cancel{2Ax^2} + \cancel{Ax^2} + \cancel{4Ax} - \cancel{4Ax} + 2A)e^x \\ &= 2Ae^x \\ &= 22e^x \end{aligned}$$

So $2A = 22$; hence $A = 11$

$$y_p = 11x^2e^x$$

General soln

$$\begin{aligned} y &= y_c + y_p \\ &= c_1e^x + c_2xe^x + 11x^2e^x \end{aligned}$$

6. [20] Use the variation of parameters method to solve the differential equation: $y'' + y = \csc^3 x$.

Homogeneous D.E.

$$y'' + y = 0$$

Auxiliary D.E.:

$$m^2 + 1 = 0$$

$$m_1 = i, \; m_2 = -i$$

$y_c = c_1 \cos x + c_2 \sin x$, $c_1, c_2$ = Constants

Seek a particular solution of given D.E. as $y_p = v_1(x)\cos x + v_2(x)\sin x$ with:

$$(\cos x)\, v_1' + (\sin x)\, v_2' = 0 \qquad (1)$$

$$-(\sin x)\, v_1' + (\cos x)\, v_2' = \csc^3 x \quad (2)$$

$(\sin x)\cdot(1) + (\cos x)\cdot 2$: $\underbrace{(\sin x \cos x - \cos x \sin x)}_{=0} v_1' + \underbrace{(\sin^2 x + \cos^2 x)}_{=1} v_2' = \cos x \csc^3 x$

So $v_2' = \dfrac{\cos x}{\sin^3 x} \rightarrow v_2 = \displaystyle\int \frac{\cos x\, dx}{\sin^3 x}$, Set $u = \sin x$; $du = \cos x\, dx$

$$= \int u^{-3} du$$

$$= -\frac{u^{-2}}{2} = -\frac{1}{2\sin^2 x}$$

(1) gives: $v_1' = -\dfrac{\sin x}{\cos x} v_2' = -\dfrac{\sin x}{\cos x} \cdot \dfrac{\cos x}{\sin^3 x} = -\dfrac{\sin x}{\sin^3 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

$$v_1 = \int -\csc^2 x\, dx = \cot x$$

Hence

$$y_p = \cot x \cos x - \frac{\csc x}{2}$$

General soln: $y = y_c + y_p = c_1 \cos x + c_2 \sin x + \cot x \cos x - \dfrac{\csc x}{2}$.