MAP 2302 (Differential Equations) — Answers
TEST 1, Friday February 9, 2018

Name: **PID:**

Remember that no documents or calculators are allowed during the test. You must show all your work to deserve the full credit assigned to any question. Guessed answers won't give you any credits. You may use the back of each sheet as scratch. 4 pages. Total =90 points including 10 bonus points.

1. [20] Solve the the initial-value problem:

$$\begin{cases} (1+\cos x)\dfrac{dy}{dx} - (\sin x)y = \sec^2 x \\ y(0) = 2. \end{cases}$$

D. E. may be rewritten

$$\frac{dy}{dx} - \frac{\sin x}{1+\cos x}\, y = \frac{\sec^2 x}{1+\cos x}$$

The solution formula for a linear D-E yields:

$$y = e^{\int \frac{\sin x}{1+\cos x}dx}\left[\int \frac{\sec^2 x}{1+\cos x} e^{\int -\frac{\sin x}{1+\cos x}dx} + c\right], \quad c = \text{constant}$$

$$u = 1+\cos x \qquad du = -\sin x\,dx \; ; \text{ so } \int \frac{\sin x}{1+\cos x}dx = -\int \frac{du}{u} = -\ln|u| = -\ln(1+\cos x) = \ln\frac{1}{1+\cos x}$$

$$y = e^{\ln\frac{1}{1+\cos x}}\left[\int \frac{\sec^2 x}{1+\cos x} e^{\ln(1+\cos x)}dx + c\right]$$

$$= \frac{1}{1+\cos x}\left[\int \frac{\sec^2 x\,\cancel{(1+\cos x)}}{\cancel{1+\cos x}}dx + c\right]$$

$$= \frac{1}{1+\cos x}\left[\int \sec^2 x\,dx + c\right]$$

$$= \frac{1}{1+\cos x}\left[\tan x + c\right]$$

Now $y(0) = \frac{1}{1+1}[0+c] = \frac{c}{2} = 2 \to c = 4$

Solution of IVP: $y = \dfrac{1}{1+\cos x}(\tan x + 4)$

2. [20] a) Reduce the equation $(2x - 3y + 7)dx + (x + y + 1)dy = 0$ to a homogeneous equation. Do not solve the homogeneous equation obtained.

Solve the linear system

$$2h - 3k = -7 \quad (1)$$
$$h + k = -1 \quad (2)$$

(1) + 3·(2) yields:

$$2h - \cancel{3k} + 3h + \cancel{3k} = -7 - 3$$
$$5h = -10 \rightarrow h = -2$$

(1) − 2·(2) yields

$$\cancel{2h} - 3k - \cancel{2h} - 2k = -7 + 2 = -5$$
$$-5k = -5 \rightarrow k = +1$$

Set $x = u - 2$
$y = v + 1$

Then $dx = du$, $dy = dv$

The D-E becomes

$$(2(u-2) - 3(v+1) + 7)\,du + (u - 2 + v + 1 + 1)\,dv = 0$$

or $(2u - 3v)du + (u + v)dv = 0$

which is homogeneous.

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b) Find all values of p such that $f(x) = e^{px}$ solves the differential equation: $y''' + y'' - 3y' - 3y = 0$. Do not solve the differential equation.

$$f'(x) = pe^{px}, \quad f''(x) = p^2 e^{px}, \quad f'''(x) = p^3 e^{px}$$

$$f'''(x) + f''(x) - 3f'(x) - 3f(x) = (p^3 + p^2 - 3p - 3)e^{px} = 0$$

So $p^3 + p^2 - 3p - 3 = 0$, as $e^{px} \neq 0$ for all x

Now $p^3 + p^2 - 3p - 3 = p^2(p+1) - 3(p+1) = (p^2 - 3)(p+1) = (p - \sqrt{3})(p + \sqrt{3})(p+1)$

$(p - \sqrt{3})(p + \sqrt{3})(p+1) = 0 \rightarrow p = -\sqrt{3}$, $p = \sqrt{3}$ or $p = -1$

3. [20] Find the orthogonal trajectories to the family of curves $y = cx^4$.

D.E for family: $\frac{dy}{dx} = 4cx^3 = \frac{4}{x} cx^4 = \frac{4y}{x}$

O.T. D.E: $\frac{dy}{dx} = -\frac{x}{4y}$

$4y\,dy = -x\,dx$

or $x\,dx + 4y\,dy = 0$, separable equ

Integrate to get

$\int x\,dx + \int 4y\,dy = c_1$ $\quad c =$ constant

$\frac{x^2}{2} + 2y^2 = c, \quad c \geq 0$

orthogonal trajectories is a family of ellipses.

4. [30] a) Show that the differential equation $3(x^2+y^2)dx + 2xydy = 0$ is not exact. b) Find an integrating factor for this equation. c) Write down the new equation and solve it.

a) $\frac{\partial}{\partial y}(3x^2+3y^2) = 6y \neq 2y = \frac{\partial}{\partial x}(2xy)$; So D.E is not exact

b) $\frac{\partial}{\partial y}\underbrace{(3x^2+3y^2)}_{M(x,y)} - \frac{\partial}{\partial x}\underbrace{(2xy)}_{N(x,y)} = 6y - 2y = 4y = \frac{2}{x}\cdot 2xy = \frac{2}{x}N(x,y)$

$$\frac{\frac{\partial M}{\partial y}(x,y) - \frac{\partial N}{\partial x}(x,y)}{N(x,y)} = \frac{2}{x}$$

Integrating factor $= e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln(x^2)} = x^2$

c) New D.E :

$(3x^2y^2 + 3x^4)dx + 2x^3ydy = 0$

Standard method $\quad \frac{\partial}{\partial y}(3x^2y^2+3x^4) = 6x^2y = \frac{\partial}{\partial x}(2x^3y)$; so D-E is exact

There exists a function F :

$\frac{\partial F}{\partial x}(x,y) = 3x^2y^2+3x^4$, (1)

$\frac{\partial F}{\partial y}(x,y) = 2x^3y$, (2)

So $F(x,y) = \int(3x^2y^2+3x^4)dx$

$= x^3y^2 + \frac{3}{5}x^5 + K(y)$ (3)

Differentiate (3) w.r.t. y :

$\frac{\partial F}{\partial y}(x,y) = 2x^3y + k'(y)$

$= 2x^3y$, by (2)

So $K'(y) = 0$, $K(y) = C$ = constant, we may choose $C=0$.

$F(x,y) = x^3y^2 + \frac{3}{5}x^5$

Solution of D.E : $x^3y^2 + \frac{3}{5}x^5 = C$, C = constant

Grouping method : $3x^2y^2dx + 2x^3dy + 3x^4dx = 0$

$y^2d(x^3) + x^3d(y^2) + d(\frac{3x^5}{5}) = 0$

$d(x^3y^2 + \frac{3x^5}{5}) = 0$

$x^3y^2 + \frac{3x^5}{5} = C$, C = constant.