MAS 3105 (Linear Algebra)
Test 1, Friday May 23, 2014

Name:

PID:

Remember that you won't get any credit if you do not show the steps to your answers. You may show your work on the back of each page.

1. [12] Find the reduced row echelon form of the matrix $B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[-2r_2+r_3]{-4r_1+r_4}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & -9 \end{pmatrix} \xrightarrow{-3r_2+r_4} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{2r_2+r_1} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-r_2} \underbrace{\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{RREF of } B}$$

2. [20] State whether each of the following statement is true or false. No explanations needed.

a) If A and B are $n \times n$ matrices, then $(A+B)^2 = A^2 + 2AB + B^2$; False, since $AB \neq BA$ in general

b) If A is a 20×20 matrix , then A is nonsingular. False; A ~~might~~ not be row equivalent to I_{20}

c) If H is a 23×23 matrix, then $\det(-5H) = -5\det(H)$. False; $\det(-5H) = (-5)^{23}\det H$

d) If S is a nonempty subset of a vector space V, then S is a subspace of V. False; S must be closed under addition and scalar multiplication

e) If U, and W are subspaces of $\mathbb{R}^n$, then $U \cap W$ is a subspace of $\mathbb{R}^n$. True (see Pb 20, section 3.1)

The zero vector is always a solution → f) Every homogeneous linear system is consistent. True (see Proof of Theorem 1.2.1)

g) If $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ is a 4×4 matrix with $\mathbf{a}_1 - 2\mathbf{a}_2 + 5\mathbf{a}_3 = \mathbf{0}$, then A is singular. True; $A \begin{pmatrix} 1 \\ -2 \\ 5 \\ 0 \end{pmatrix} = 0_{\mathbb{R}^4}$ & Th 1.5.2

h) If A and B are $n \times n$ matrices, then $\det((AB)^T) = \det(A)\det(B)$. True; $\det((AB)^T) = \det(B^T)\det(A^T) = \det(B)\det(A) = \det(A)\det(B)$

i) If $\mathbf{x}$ is a nonzero vector in $\mathbb{R}^6$ and A is a 6×6 matrix with $A\mathbf{x} = \mathbf{0}$, then $\det(A) = 0$. True; Theorem 1.5.2 & Theorem 2.2.2

j) If A and B are 15×15 matrices with $\det(A) = \det(B)$, then $A = B$. False

Pick $A = I_{15}$

B = matrix obtained from A by changing r_2 to $2r_1 + r_2$.

$\det A = \det B$, and $A \neq B$.

3. [20] Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 2 \\ -1 & 0 & -1 \end{pmatrix}$.

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{6} & -\frac{7}{6} \\ 0 & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 6 & 2 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_1+r_3]{-2r_1+r_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -4 & -2 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{2r_3+r_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 6 & 0 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-r_3+r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 6 & 0 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{-\frac{r_2}{3}+r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 6 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & -\frac{1}{3} & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow{-\frac{r_3}{2}+r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{6} & -\frac{7}{6} \\ 0 & 6 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & -\frac{1}{3} & \frac{1}{3} \end{array}\right) \xrightarrow[r_3/2]{r_2/6} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{6} & -\frac{7}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{6} \end{array}\right)$$

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4. [25] a) Let A be an $m \times n$ matrix. Show that A^TA and AA^T are both symmetric.

$(A^TA)^T = A^T(A^T)^T = A^TA$, by transposition rules; so A^TA is symmetric

$(AA^T)^T = (A^T)^TA^T = AA^T$, — " — " —; so AA^T is symmetric

Note: $m=n$ was to avoid any confusion; properties hold for all m and n.

b) Let A and B are $n \times n$ matrices with $A\mathbf{y} = B\mathbf{y}$ for some nonzero vector $\mathbf{y}$ in $\mathbb{R}^n$. Show that $\det(A-B)=0$.

$Ay = By \Rightarrow Ay - By = 0_{\mathbb{R}^n} \Rightarrow (A-B)y = 0_{\mathbb{R}^n}$ & $y \neq 0_{\mathbb{R}^n}$; so $A-B$ is singular by Theorem 1.5.2. Therefore $\det(A-B)=0$ by Theorem 2.2.2.

c) Let $C = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \\ 2 & 5 & 6 \end{pmatrix}$. Find an elementary matrix E such that $D = EC$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix} \xrightarrow{2r_1+r_2} \begin{pmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \\ 2 & 5 & 6 \end{pmatrix};\ \text{consequently} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2r_1+r_2} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

Check $EC = D$.

d) Compute $\det(C)$ with C as in question c).

$$\det C = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 5 & 6 \end{vmatrix} = 1\,(3(6)-5(4)) - 1\,(2(6)-2(4)) + 1\,(2(5)-2(3))$$
$$= -2 - 4 + 4$$
$$= -2.$$

5. [5] Let V be a vector space, and let S be a nonempty subset of V. Complete the sentence: S is called a subspace of V when

(i) $x+y \in S$ whenever $x, y \in S$

(ii) $\alpha x \in S$ whenever α is a scalar and $x \in S$.

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6. [8] Let A be a 23×23 matrix with $A^T = -A$. Show that A is singular. (Hint. You may use determinants.)

$\det(A^T) = \det(-A) = (-1)^{23} \det A = -\det A$, and
$\det(A^T) = \det A$; so $\det A = -\det A$ or $2 \det A = 0$; hence $\det A = 0$, and A is singular by Theorem 2.2.2.

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7. [10] Let A be an $n \times n$ matrix. Set $S = \{A\mathbf{x} \in \mathbb{R}^n; \mathbf{x} \in \mathbb{R}^n\}$. Show that S is a subspace of the vector space $(\mathbb{R}^n, +, .)$, where . stands for scalar multiplication. $0_{\mathbb{R}^n} = A 0_{\mathbb{R}^n}$ so $0_{\mathbb{R}^n} \in S$, and $S \neq \emptyset$.

Let $u \in S$, $v \in S$; then $u = Ax_1$, and $v = Ax_2$, $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^n$

$u + v = Ax_1 + Ax_2 = A(x_1 + x_2)$ with $x_1 + x_2 \in \mathbb{R}^n$; so $u + v \in S$

Let α be a scalar. Then $\alpha u = \alpha(Ax_1) = A(\alpha x_1)$; and $\alpha x_1 \in \mathbb{R}^n$ so $\alpha u \in S$. Hence S is a subspace of $\mathbb{R}^n$.

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8. [10] Let A be an $n \times n$ matrix. Use mathematical induction to prove that for every integer $m \geq 2$, one has $\det(A^m) = (\det(A))^m$.

Basis step: $m = 2$. $\det(A^2) = \det(A.A) = \det A . \det A = (\det A)^2$, by Theorem 2.2.3
property holds for $m = 2$

Induction hypothesis: Let $m \geq 3$. Suppose that $\det(A^{m-1}) = (\det(A))^{m-1}$.
Show that $\det(A^m) = (\det(A))^m$.

$$\begin{aligned}
\det(A^m) &= \det(A^{m-1}A) \\
&= \det(A^{m-1}) \det A, \text{ by Theorem 2.2.3} \\
&= (\det(A))^{m-1} \det A, \text{ by Induction hypothesis} \\
&= (\det(A))^{m-1+1} \\
&= (\det A)^m.
\end{aligned}$$

Hence property holds for all $m \geq 2$.

3. Another method for finding A^{-1} is by using adj$A$.

$$A^{-1} = \frac{\operatorname{adj} A}{\det A}, \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 2 \\ -1 & 0 & -1 \end{pmatrix} \xrightarrow[r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -4 \\ 0 & 2 & 2 \end{pmatrix}$$

So $\det A = 1(2(2)+2(4)) = 12$

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 6 & 2 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 6 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 2 & -14 \\ 0 & +2 & 4 \\ 6 & -2 & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -\frac{6}{12} & \frac{2}{12} & -\frac{14}{12} \\ 0 & \frac{2}{12} & \frac{4}{12} \\ \frac{6}{12} & -\frac{2}{12} & \frac{2}{12} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{6} & -\frac{7}{6} \\ 0 & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}$$