MAS 3105 (Linear Algebra) — Key
Test 1, Friday May 22, 2015

Name: PID:

Remember that you won't get any credit if you do not show the steps to your answers. You may show your work on the back of each page. There are 15 bonus points which do not carry over to other assignments or exams.

1. [20] a) Find the reduced row echelon form of the matrix $B = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{pmatrix}$. b) Find the null space $N(B)$ of B, and specify a spanning set for $N(B)$.

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{pmatrix} \xrightarrow[-5r_1+r_3]{-3r_1+r_2} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{pmatrix} \xrightarrow{-2r_2+r_3} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{r_2}{4}}$$

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-3r_2+r_1} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{RREF of } B$$

b) $N(B) = \{x \in \mathbb{R}^4;\ Bx = 0_{\mathbb{R}^3}\}$. If $x = (x_1, x_2, x_3, x_4)^T$, $Bx = 0$ is equivalent to $Ux = 0_{\mathbb{R}^3}$ where U = RREF of B. So $x_2 + 2x_3 + 3x_4 = 0 \to x_2 = -2x_3 - 3x_4$, and $x_1 - x_3 - 2x_4 = 0 \to x_1 = x_3 + 2x_4$. Hence $N(B) = \{(\alpha+2\beta, -2\alpha-3\beta, \alpha, \beta)^T;\ \alpha, \beta \in \mathbb{R}\}$

$\forall x \in B$, $x = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$, $\alpha, \beta \in \mathbb{R}$. So $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a spanning set for $N(B)$.

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2. [20] State whether each of the following statement is true or false. No explanation needed.

a) If A and B are $n \times n$ matrices, then $(A - B)(A + B) = A^2 - B^2$. False, $AB \neq BA$ in general

b) If S is a nonempty subset of a vector space V, then S contains the null vector of V. False

c) If H is a 11×17 matrix, then H^T is a 17×11 matrix. True by defn of transpose

d) If A is a singular 22×22 matrix , then A^T is singular. True, by Theorem 2.1.2 & Theorem 2.2.2

e) If U, and W are subspaces of $\mathbb{R}^n$, then $U \cup W$ is a subspace of $\mathbb{R}^n$. False. See example solved in class

f) If A and B are $n \times n$ singular matrices, then $A + B$ is also singular. False

g) If $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is a 4×3 matrix with $N(A) = \{\mathbf{0}\}$, and $\mathbf{b} = -3\mathbf{a}_1 + 4\mathbf{a}_2 - 7\mathbf{a}_3$, then $A\mathbf{x} = \mathbf{b}$ has a unique solution. True

h) If A and U are $n \times n$ matrices and U is nonsingular, then $\det(U^{-1}AU) = \det(A)$. True, by Th. 2.2.3 & $\det(U^{-1}) = \frac{1}{\det U}$

i) If A is a 6×5 matrix and B is a 5×6 matrix, then the product AB is a 6×6 matrix. True by defn of product

j) If A is a 12×12 matrix and $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$, then A is nonsingular. True, by Th. 1.5.2

Let $n \geq 2$.
Let $p < n$. Set $A = E_{11} + \cdots + E_{pp}$, $B = E_{p+1\,p+1} + \cdots + E_{nn}$. Then $A + B = I_n$, while both A and B are singular. Remember E_{ij} = matrix with 1 as entry at i-th row and j-th column, and zero entry everywhere else.

3. [20] Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-3r_1+r_3]{-2r_1+r_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 3 & -3 & 0 & 1 \end{array}\right) \xrightarrow{-r_2+r_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow[-2r_3+r_2]{-r_2+r_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 0 & 0 & +3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow{r_3+r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Hence $A^{-1} = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$

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4. [25] a) Let A be a nonsingular $n \times n$. Show that $A^T A$ is nonsingular and $\det(A^T A) > 0$.

$\det(A^TA) = \det(A^T)\det(A) = (\det(A))^2$. Since A is nonsingular, $\det A \neq 0$. So $\det(A^TA) \neq 0$, and A^TA is nonsingular with $\det(A^TA) > 0$.

b) Let A be an $n \times n$ matrix with $\det(A - 22I_n)=0$. Show that the system $Ax = 22x$ has a nontrivial solution.

$Ax = 22x$ is equivalent to $(A - 22I_n)x = 0_{\mathbb{R}^n}$. Now $A - 22I_n$ is singular, by Theorem 2.2.2, as $\det(A - 22I_n) = 0$. Hence $Ax = 22x$ has a nontrivial solution by Theorem 1.5.2.

c) Let $C = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}$. Find the LU factorization of C.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\xrightarrow{-2r_2+r_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = U, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}, \quad LU = C$$

d) Compute $\det(C)$ with C as in question c).

$\det(C) = \det(LU) = \det(L)\det(U) = 1(-2) = -2$

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3. Method 2: using $\operatorname{adj} A$: $A^{-1} = \operatorname{adj} A / \det A$. $\det A = (18-16) - (12-12) + (8-9) = 1$

$$\operatorname{adj} A = \begin{pmatrix} \begin{vmatrix} 3 & 4 \\ 4 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 4 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} = A^{-1}, \text{ as } \det(A) = 1$$

5. [5] Let E be a vector space, and let U be a nonempty subset of E. Complete the sentence: U is called a subspace of E when

i) $x+y \in U$ whenever $x \in U$ and $y \in U$

ii) $\alpha x \in U$ whenever $x \in U$, and α is a scalar.

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6. [5] Prove that any two nonsingular $n \times n$ matrices are row equivalent. Let A, B be two nonsingular matrices. Then A is row equivalent to I_n and B is row equivalent to I_n. So $A = E_k \cdots E_1 I_n$ for some elementary matrices $E_k, \ldots, E_1$. Similarly $B = F_\ell \cdots F_1 I_n$ — " — " — " — " — " $F_1, \ldots, F_\ell$. so $I_n = F_1^{-1} F_2^{-1} \cdots F_\ell^{-1} B$, and $A = E_k E_{k-1} \cdots E_1 F_1^{-1} F_2^{-1} \cdots F_\ell^{-1} B$; hence A and B are row equivalent.

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7. [10] Let U and V be subspaces of a vector space E. Set $W = \{z \in E; z = u + v \text{ with } u \in U \text{ and } v \in V\}$. Show that W is a subspace of E. Since $0_E \in U$, $0_E \in V$ and $0_E = 0_E + 0_E$; it follows that $0_E \in W$; $W \neq \phi$. Let $y \in W$, $z \in W$. Show $y+z \in W$. $y = u_1 + v_1$, $u_1 \in U$, $v_1 \in V$, $z = u_2 + v_2$, $u_2 \in U$, $v_2 \in V$; so $y+z = u_1+v_1+u_2+v_2 = (u_1+u_2)+(v_1+v_2) \in U+V$ as $u_1+u_2 \in U$, and $v_1+v_2 \in V$; So $y+z \in W$. Let α be a scalar. Show $\alpha y \in W$.

$\alpha y = \alpha(u_1+v_1) = \alpha u_1 + \alpha v_1 \in U+V$, Since $\alpha u_1 \in U$ and $\alpha v_1 \in V$; so $\alpha y \in W$.

Hence W is a subspace of E.

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8. [10] Use mathematical induction on m to prove that for every integer $m \geq 2$, if $A_1, A_2, \ldots, A_m$ are $n \times n$ matrices, then one has $(A_1 A_2 \ldots A_m)^T = A_m^T \ldots A_2^T A_1^T$.

Basis Step: $m = 2$. Show $(A_1 A_2)^T = A_2^T A_1^T$. $(A_1 A_2)^T = A_2^T A_1^T$ by transpose rule.

Inductive Step. Let $m \geq 2$. Suppose that the property holds for m. Prove it for $m+1$. In other words, show $(A_1 A_2 \cdots A_{m+1})^T = A_{m+1}^T A_m^T \cdots A_2^T A_1^T$.

Now,

$$\begin{aligned}(A_1 A_2 \cdots A_m A_{m+1})^T &= ((A_1 A_2 \cdots A_m) A_{m+1})^T \\ &= A_{m+1}^T (A_1 A_2 \cdots A_m)^T, \text{ by case } m=2 \\ &= A_{m+1}^T A_m^T \cdots A_2^T A_1^T, \text{ by inductive hyprothesis}\end{aligned}$$

Hence the property holds for all $m \geq 2$.