MAS 3105 (Linear Algebra) — key
Test 1, Friday May 27, 2016

Name: **PID:**

Remember that you won't get any credit if you do not show the steps to your answers. You may show your work on the back of each page. Total=105 points.

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1. [20] Solve the linear system

$$\begin{aligned} x_2 + x_3 + x_4 &= 0 \\ 3x_1 + 3x_2 - 4x_3 &= 9 \\ x_1 + x_2 + 2x_3 + x_4 &= 6 \\ 2x_1 + 3x_2 + x_3 + 3x_4 &= 6 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 3 & -4 & 0 & 9 \\ 1 & 1 & 2 & 1 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 6 \\ 3 & 3 & -4 & 0 & 9 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right) \xrightarrow[-2r_1+r_4]{-3r_1+r_2}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 6 \\ 0 & 0 & -10 & -3 & -9 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -6 \end{array}\right) \xrightarrow{-r_3+r_4} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 6 \\ 0 & 0 & -10 & -3 & -9 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -4 & 0 & -6 \end{array}\right) \xrightarrow[\substack{r_4 \to r_3 \\ r_3 \to r_2}]{r_2 \to r_4}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 6 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -4 & 0 & -6 \\ 0 & 0 & -10 & -3 & -9 \end{array}\right) \xrightarrow{-\frac{5}{2}r_3 + r_4} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 6 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -4 & 0 & -6 \\ 0 & 0 & 0 & -3 & 6 \end{array}\right)$$

$-3x_4 = 6 \longrightarrow x_4 = -2$

$-4x_3 = -6 \rightarrow x_3 = 3/2$

$x_2 + x_3 + x_4 = 0 \longrightarrow x_2 = -x_3 - x_4 = -\frac{3}{2} + 2 = \frac{1}{2}$

$x_1 + x_2 + 2x_3 + x_4 = 6 \longrightarrow x_1 = -x_2 - 2x_3 - x_4 + 6 = -\frac{1}{2} - 3 + 2 + 6 = \frac{9}{2}$

$S = \left\{ \left(\frac{9}{2}, \frac{1}{2}, \frac{3}{2}, -2\right)^T \right\}$

2. [20] State whether each of the following statement is true or false.
No explanations needed.

(1) If A is an $n \times n$ matrix with $A^2 = 0_{\mathcal{M}_n}$, then A is singular. True, $\det(A^2) = 0 = (\det A)^2$; so $\det A = 0$

(2) If A a 20×20 matrix that is row equivalent to a nonsingular matrix B, then $\det(A) \neq 0$. True, $A = E_p E_{p-1} \cdots E_1 B$; $E_p, \dots, E_1$ are elementary matrices. A is nonsingular as a product of nonsingular matrices

(3) If A is an 11×15 matrix, then $A^T A$ is a 15×15 matrix. True, by the defn of A^T and product

(4) If U is a nonempty subset of a vector space E, then U is a subspace of E. False, U must be closed under addition and scalar multiplication

(5) If A, and B satisfy $\det(A) = \det(B)$, then $\det(AB) \geq 0$. True, $\det(AB) = (\det A)^2$

(6) If $A^2 - 3A + I_n = 0_{\mathcal{M}_n}$, then A is nonsingular. True $I_n = A(3I_n - A)$

(7) If an $m \times m$ matrix A satisfies $A^T = A$, then A is nonsingular. False, pick $A = 0_{\mathcal{M}_m}$

(8) If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$. True $\det(AB) = \det A \cdot \det B = \det(BA)$

(9) If A and B are nonsingular $n \times n$ matrices, then $A + B$ is also nonsingular. False, pick $A = I_n$, $B = -I_n$

(10) If A and B are 15×15 matrices with $A = B^T$, then $\det(A) = \det(B)$. True $\det(A) = \det(B^T) = \det(B)$

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3. [15] Consider the linear system whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & m & 3 & 0 \\ -1 & 1 & 5 & 0 \end{array}\right]$

a) Is it possible for this system to be inconsistent? Explain, or no credit.

This system cannot be inconsistent since $x = 0_{\mathbb{R}^3}$ is a solution.

b) For which value(s) on m will the system have infinitely many solutions? Write down the solution set(s) in this case.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & m & 3 & 0 \\ -1 & 1 & 5 & 0 \end{array}\right) \xrightarrow[r_1+r_3]{-2r_1+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & m-4 & 1 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right) \xleftrightarrow{\frac{r_3}{3} \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & m-4 & 1 & 0 \end{array}\right) \xrightarrow{(4-m)r_2+r_3}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 9-2m & 0 \end{array}\right)$$

We have $(9-2m)x_3 = 0$. If $m \neq 9/2$, then $x_3 = 0$, $x_2 + 2x_3 = 0 \to x_2 = 0$, $x_1 + 2x_2 + x_3 = 0 \to x_1 = 0$; so only solution is $x = 0_{\mathbb{R}^3}$ if $m \neq 9/2$.
If $m = 9/2$, then $x_2 + 2x_3 = 0$, so $x_2 = -2x_3$
$x_1 + 2x_2 + x_3 = 0 \to x_1 = -2x_2 - x_3 = 3x_3$, and x_3 is arbitrary; $S = \{(3\alpha, -2\alpha, \alpha)^T; \alpha \in \mathbb{R}\}$.
So the system has infinitely many solutions when $m = 9/2$.

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4. [10] For which values of the number a do we have $A_a^2 = I_2$ if $A_a = \begin{bmatrix} a-1 & 1 \\ -2 & 1-a \end{bmatrix}$, and I_2 denotes the identity matrix of order 2?

$$A_a^2 = \begin{pmatrix} (a-1)^2 - 2 & \underbrace{a-1+1-a}_{=0} \\ \underbrace{-2(a-1) - 2(1-a)}_{=0} & \underbrace{-2 + (1-a)^2}_{=(a-1)^2 - 2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ iff}$$

$(a-1)^2 - 2 = 1 \to (a-1)^2 = 3 \to a = 1 \pm \sqrt{3}$.

$$\operatorname{adj} A = \begin{pmatrix} \begin{vmatrix} 3 & 5 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 5 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 2 \\ -5 & 4 & -3 \\ 3 & -2 & 1 \end{pmatrix}$$

$\det A = 1(0) - 1(5) + 1(3) = -2$

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} = \frac{\operatorname{adj} A}{-2}$$

5. [20] Find the inverse of the matrix (Hint. You may use the reduction method.)

$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 5 \end{pmatrix}$. We start we the augmented matrix $(A \mid I_3)$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-r_1+r_3]{-2r_1+r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right) \xrightarrow{-2r_2+r_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 3 & -2 & 1 \end{array}\right) \xrightarrow[-r_2+r_1]{-r_3/2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 3 & -1 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & -\frac{1}{2} \end{array}\right) \xrightarrow[-3r_3+r_2]{2r_3+r_1}$$

U

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & \frac{5}{2} & -2 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & -\frac{1}{2} \end{array}\right)$$

Hence $A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ \frac{5}{2} & -2 & \frac{3}{2} \\ -\frac{3}{2} & 1 & -\frac{1}{2} \end{pmatrix}$

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6. [10] a) Let A be an $n \times n$ matrix. Set $B = A + A^T$ and $D = A^T - A$. Show that B is symmetric and D is skew symmetric.

$B^T = (A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B$; so B is symmetric

$D^T = (A^T - A)^T = (A^T)^T - A^T = A - A^T = -(A^T - A) = -D$; So D is skew symmetric

b) Let A denote the matrix in problem 5. Find an upper triangular matrix U and a lower triangular matrix L such that $A = LU$.

From Pb 5, $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$

$LU = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 5 \end{pmatrix}$, by definition of product; so $A = LU$

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7. [10] a) Let V be a vector space, and let S be a subset of V. Complete the sentence: S is called a subspace of V when S is nonempty and further satisfies

i) $\alpha x \in S$ for all scalar α and all x in S

ii) $x + y \in S$ for all x, y in S.

b) Let A and B be $m \times m$ matrices with $AB = A + B$. Show that, if B is nonsingular, then A is nonsingular.

If B is nonsingular, then B^{-1} exists; so $\underbrace{(AB)B^{-1}}_{=A} = (A+B)B^{-1} = AB^{-1} + BB^{-1} = AB^{-1} + I_m$; so

$A = AB^{-1} + I_m$; hence $A - AB^{-1} = I_m$ or

$A(I_m - B^{-1}) = I_m$; So A is nonsingular.