MAS 3105 (Linear Algebra)
Test 2, Friday June 05, 2015

Name: **PID:**

Remember that you won't get any credit if you do not show the steps to your answers. Total=105 points.

1. [20] Find a basis for the null space and a basis for the column space of the matrix $A = \begin{pmatrix} -1 & 2 & -3 & 5 \\ 2 & 3 & 7 & 1 \\ 6 & 16 & 22 & 14 \end{pmatrix}$.

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2. [20] State whether each of the following statement is true or false. No explanations needed.

a) In $\mathbb{R}^3$, there exist four linearly independent vectors.

b) If $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are linearly independent in $\mathbb{R}^5$, then they span $\mathbb{R}^5$.

c) If A is a 7×5 matrix , then A and A^T have the same rank.

d) If U is the reduced row echelon form of a nonsingular matrix A, then A and U have the same column space.

e) If If $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ span a subspace of $\mathbb{R}^4$, then they are linearly independent.

f) If A is a 12×12 nonsingular matrix , then A and A^T have the same nullity.

g) If L is a linear operator on $\mathbb{R}^n$ with $\ker(L) = \{0_{\mathbb{R}^n}\}$, then $R(L) = \mathbb{R}^n$.

h) If U and V are subspaces of a vector space E, then $U + V$ is a subspace of E too.

i) If A and B are similar matrices and A is singular, then B is also singular.

j) If A and B are similar matrices, then A^T and B^T are also similar matrices.

3. [10] Let $\mathcal{M}_2 = \left\{ A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}; a,\ b,\ c,\ d \in \mathbb{R} \right\}$ be the space of 2×2 matrices. Define on $\mathcal{M}_2$ a mapping L by $L(A) = A + A^T$. a) Show that L is linear. b) Find a basis for $\ker(L)$ and a basis for $\mathrm{R}(L)$.

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4. [25] Let $\mathbf{v}_1 = (-1, -2, 1)^T$, $\mathbf{v}_2 = (1, 3, 2)^T$, $\mathbf{v}_3 = (1, 1, 2)^T$, and $\mathbf{w}_1 = (-1, -3, 1)^T$, $\mathbf{w}_2 = (2, 3, 1)^T$ and $\mathbf{w}_3 = (1, 1, 3)^T$ be vectors in $\mathbb{R}^3$. Let L be the linear operator defined on $\mathbb{R}^3$ by

$$L(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) = (-x_1 + x_2 + 2x_3)\mathbf{v}_1 + (x_1 + 2x_2 - x_3)\mathbf{v}_2 + (2x_1 - x_2 + x_3)\mathbf{v}_3.$$

a) Find the matrix representation M of L relative to the ordered basis $B = [\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3]$. b) Find the transition matrix T from the ordered basis B to the ordered basis $D = [\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3]$. c) Write down the matrix P of L with respect to the ordered basis D in terms of M, but do not attempt to find the entries of P. d) If $\mathbf{u} = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$, find the coordinates of $L(\mathbf{u})$ in the ordered basis D.

5. [15] a) Let A and B be 6×9 matrices. If rank of A is 5, what is the dimension of $N(A)$? If the dimension of $N(B)$ is 6, what is the rank of B? (Explain each answer to get full credit.)

b) Use the Wronskian to show that the vectors $1,\ e^x - e^{-x},\ e^x + e^{-x}$ are linearly independent in $C^2([0,1])$.

c) Complete the sentence: The vectors $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_n$ form a basis for $\mathbb{R}^n$ when the following two conditions are met:

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6. [10] Let $\mathbf{u}_1 = (-1, 1, 1)^T$, $\mathbf{u}_2 = (2, a, 2)^T$ and $\mathbf{u}= (-1, a^2+2, 5)^T$ be vectors in $\mathbb{R}^3$. For which values of a does the vector $\mathbf{u}$ belong to Span$(\mathbf{u}_1, \mathbf{u}_2)$?

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7. [5] Let L denote a linear operator on a vector space E. Let U denote a subspace of E. Set $L^{-1}(U) = \{v \in E; L(v) \in U\}$. Show that $L^{-1}(U)$ is a subspace of E.