MAC 2311 (Calculus I) — Key
TEST 2, Friday November 13, 2009

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good Luck.

1. [20] Evaluate the following limits (Show all your work)

a) $\lim_{x\to 0} \frac{\sin(x)}{e^x - 1} = \lim_{x\to 0} \frac{\cos(x)}{e^x}$, by HR
$= \frac{1}{1}$
$= 1$

b) $\lim_{x\to+\infty} (\ln(2+x) - \ln(3+2x)) = \lim_{x\to+\infty} \ln\left(\frac{2+x}{3+2x}\right)$
$= \ln\left(\lim_{x\to+\infty} \frac{2+x}{3+2x}\right)$
$= \ln\left(\lim_{x\to+\infty} \frac{1}{2}\right)$, by HR
$= \ln\left(\frac{1}{2}\right) = -\ln(2)$

c) $\lim_{x\to 0} \frac{x^3}{x - \tan(x)} = \frac{0}{0}$
$= \lim_{x\to 0} \frac{3x^2}{1-\sec^2 x}$, by HR
$= \lim_{x\to 0} \frac{6x}{-2\sec^2 x \tan x} = -3 \lim_{x\to 0} \frac{1}{2\sec^2 x \tan^2 x + \sec^4 x} = -3\left(\frac{1}{1}\right) = -3$

d) $\lim_{x\to 0^+} (\sin(2x))^x = e^{\lim_{x\to 0^+} \ln \sin(2x)^x}$
$= e^{\lim_{x\to 0^+} x \ln(\sin(2x))} = e^{\lim_{x\to 0^+} \frac{\ln(\sin(2x))}{1/x}} = e^{\lim_{x\to 0^+} \frac{2\cos(2x)}{\sin(2x)} \cdot \frac{1}{-1/x^2}}$, by HR
$= e^{\lim_{x\to 0^+} -\frac{2x}{\sin(2x)} \cdot x\cos(2x)}$
$= e^{-1(0)}$
$= e^0 = 1$

2. [8] Use an appropriate local linear approximation to estimate the value of $\sqrt[3]{26.46}$.

Set $f(x) = \sqrt[3]{x}$, $x_0 = 27$; $f'(x) = \frac{1}{3}x^{-2/3}$

$f(x) \approx f(x_0) + (x - x_0)f'(x_0)$, for x close to x_0.

$f(26.46) \approx \sqrt[3]{27} + (26.46 - 27)\cdot\frac{1}{3}(27)^{-2/3} = 3 - (0.54)\left(\frac{1}{3}\cdot\frac{1}{9}\right)$

$\approx 3 - \frac{0.54}{27} = 3 - 0.02 = 2.98$

3. [8] A point P is moving along the curve $2y - x^3 = 2$. When P is at (2,5), y is increasing at the rate of 2 units/s. How fast is x changing?

$$\frac{d}{dt}(2y - x^3) = \frac{d}{dt}(2) = 0$$

$$2\frac{dy}{dt} - 3x^2\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \frac{2}{3x^2}\frac{dy}{dt}$$

$$\left.\frac{dx}{dt}\right|_{\substack{x=2\\y=5}} = \frac{2}{3(2^2)}(2) = \frac{1}{3}$$; x is increasing at the rate of $\frac{1}{3}$ unit/s

4. [10] a) State the Mean-value theorem. b) Show that the function f defined by $f(x) = x^3 + x - 4$, x in $[-1, 2]$, satisfies all the requirements of the Mean Value Theorem. c) Find all numbers x_0 in $(-1, 2)$ such that $f'(x_0) = \frac{f(2)-f(-1)}{2-(-1)}$.

a) See Notes or textbook.

b) f is a polynomial so f is continuous on $[-1,2]$, and differentiable on $(-1,2)$; f satisfies all the requirements of the MVT. c) There exists x_0 in $(-1,2)$ with $f'(x_0) = \frac{f(2)-f(-1)}{3} = \frac{6-(-6)}{3} = 4$; Now $f'(x) = 3x^2+1$, so $3x_0^2+1 = 4$; $3x_0^2 = 3$; $x_0^2 = 1$

$x_0 = -1$ or $x_0 = 1$; but only $x_0 = 1$ lies in $(-1,2)$.

5. [20] Find the derivative of each of the following functions (Show all your work, and simplify your answers as much as possible; you will not get any credit by guessing the correct answer(s).)

a) $f(x) = \cos(e^x)$

$$f'(x) = e^x(-\sin(e^x))$$
$$= -e^x\sin(e^x)$$

b) $g(x) = e^{(x \sin x)}$

$$g'(x) = (\sin x + x\cos x)\,e^{(x\sin x)}$$

c) $h(x) = \tan^{-1}(x^2 - x)$

$$h'(x) = (2x-1)\cdot\frac{1}{1+(x^2-x)^2}$$
$$= \frac{2x-1}{1+x^2(x-1)^2}$$

d) $k(x) = \sin^{-1}(\ln x)$

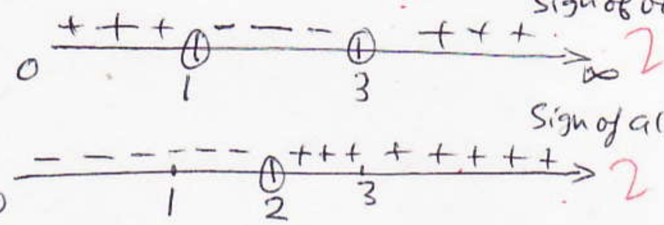
$$k'(x) = \frac{1}{x}\cdot\frac{1}{\sqrt{1-(\ln x)^2}}$$
$$= \frac{1}{x\sqrt{1-(\ln x)^2}}$$

6. [14] The function $s(t) = t^3 - 6t^2 + 9t + 1, \quad t \geq 0$, describes the position of a particle moving along a straight line, where s is in feet and t is in seconds. a) Find the velocity and acceleration functions. b) At what times is the particle stopped? c) When is the particle speeding up? Slowing down? d) Give a schematic picture of the motion.

a) $v(t) = s'(t) = 3t^2 - 12t + 9$ | , $a(t) = v'(t) = 6t - 12$ |

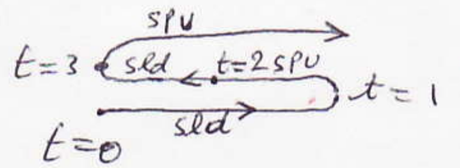
b) Particle is stopped when $v(t) = 0$; $3(t^2 - 4t + 3) = 0$; $3(t-1)(t-3) = 0$
$t = 1$ or $t = 3$. 2

c) Particle is speeding up when velocity and acceleration have the same sign, and slowing down otherwise.



Particle is speeding up on $(1, 2) \cup (3, \infty)$ 2
and slowing down on $(0, 1) \cup (2, 3)$ 2

d)



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7. [20] Evaluate each indefinite integral. (Show all your work)

a) $\int \left(\frac{-2x^{13} + x^3 - \frac{7}{\sqrt[11]{x}} + 5}{x^4} \right) dx = -2\int x^9 dx + \int \frac{dx}{x} - 7\int x^{-\frac{1}{11} - 4} dx + 5\int x^{-4} dx$

$= -\frac{2x^{10}}{10} + \ln|x| - 7\frac{x^{-34/11}}{-34/11} + 5\frac{x^{-3}}{-3} + C$

$= -\frac{x^{10}}{5} + \ln|x| + \frac{77}{34}x^{-\frac{4}{11}} - \frac{5}{x^3} + C$

b) $\int (x^6 + e^x)^{1113}(6x^5 + e^x)\, dx = \int u^{1113} du$

$u = x^6 + e^x$
$du = (6x^5 + e^x)dx$

$= \frac{u^{1114}}{1114} + C$

$= \frac{(x^6 + e^x)^{1114}}{1114} + C$

c) $\int \cos x \cos(\sin x)\, dx = \int \cos(u) du$

$u = \sin x$
$du = \cos x\, dx$

$= \sin(u) + C$

$= \sin(\sin x) + C$

d) $\int \sec x(\tan x + \sec x)\, dx = \int \sec x \tan x\, dx + \int \sec^2 x\, dx$

$= \sec x + \tan x + C$

Bonus. [6] $\int \frac{x^4 + 2x^2 + 3}{x^2 + 1}\, dx = \int \frac{(x^4 + 2x^2 + 1) + 2}{x^2 + 1} dx$

$= \int \frac{(x^2+1)^2}{x^2+1} dx + 2\int \frac{dx}{1+x^2} = \int (x^2+1)dx + 2\tan^{-1}x$

$= \frac{x^3}{3} + x + 2\tan^{-1}x + C$