MAC 2311 (Calculus I) – Key
Test 2, Friday November 22, 2013

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University administration. Good Luck.

1. [10] Find the following limits.

a) $\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{x-1} = \frac{0}{0}$

$$\overset{H.R}{=} \lim_{x\to 1} \frac{-\frac{\pi}{2}\sin\frac{\pi}{2}x}{1} = -\frac{\pi}{2}\sin\frac{\pi}{2} = -\frac{\pi}{2}$$

b) $\lim_{x\to 0}(1+4x)^{-1/5x} = \lim_{x\to 0} e^{\ln[(1+4x)^{-1/5x}]}$

$$= \lim_{x\to 0} e^{-\frac{\ln(1+4x)}{5x}} = e^{-\lim_{x\to 0}\frac{\ln(1+4x)}{5x}} = e^{\frac{0}{0}}$$

$$\overset{H.R}{=} e^{-\lim_{x\to 0}\frac{4/(1+4x)}{5}} = e^{-4/5}$$

2. [30] Let $f(x) = x^2(x+4)^2$. a) Find the first and second derivatives of f. b) Find the intervals of increase and decrease. c) Find the intervals of concavity and the inflection point(s). d) List all the critical points of f, and state for each critical point whether a relative maximum, a relative minimum, or neither occurs. e) Does f have an absolute maximum? an absolute minimum?

a) $f'(x) = 2x(x+4)^2 + x^2\, 2(x+4) = 2x(x+4)(x+4+x) = 2x(x+4)(2x+4)$

$f''(x) = 2(x+4)(2x+4) + 2x(2x+4) + 2\cdot 2x(x+4) = 2(2x^2+4x+8x+16+2x^2+4x+2x^2+8x)$

$= 2(6x^2+24x+16) = 4(3x^2+12x+8)$; $f''(x)=0 \to 3(x^2+4x+4)-4=0 \to 3(x+2)^2-4=0 \to (x+2)^2=\frac{4}{3} \to x=-2\pm\frac{2}{\sqrt{3}}$

$f'(x)=0 \to x=0, x=-2, x=-4$

b) sign of $f'(x)$: $-$ ($x=-5$), $+$ ($x=-3$), $-$ ($x=-1$), $+$ ($x=1$); on $-\infty$, -4, -2, 0, $+\infty$

- f is increasing on $[-4,-2]\cup[0,+\infty)$
- f is decreasing on $(-\infty,-4]\cup[-2,0]$

c) sign of $f''(x)$: $+$, $-$, $+$; on $-\infty$, $-2-\frac{2}{\sqrt{3}}$, $-2+\frac{2}{\sqrt{3}}$, $+\infty$

- f is CU on $(-\infty,-2-\frac{2}{\sqrt{3}})\cup(-2+\frac{2}{\sqrt{3}},+\infty)$
- f is CD on $(-2-\frac{2}{\sqrt{3}}, -2+\frac{2}{\sqrt{3}})$
- Inflection points: $(-2-\frac{2}{\sqrt{3}}, f(-2-\frac{2}{\sqrt{3}}))$, $(-2+\frac{2}{\sqrt{3}}, f(-2+\frac{2}{\sqrt{3}}))$

d) C.Ps: $x=0$, $x=-2$, $x=-4$

- f has a relative minimum of 0 at $x=-4$ and $x=0$
- f has a relative maximum of 16 at $x=-2$, by FDT

e) $f(x)\geq 0$ for all x; so f has an absolute minimum of 0 at $x=0$ & $x=-4$. $\lim_{x\to+\infty} f(x) = +\infty$; f has no absolute maximum

3. [10] Find an appropriate local linear approximation and use it to approximate $\sqrt[4]{15.68}$.

Set $f(x) = \sqrt[4]{x} = x^{1/4}$. Then $f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$. Set $x_0 = 16$

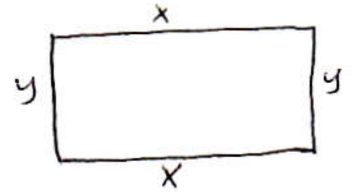
$$\sqrt[4]{15.68} \approx \sqrt[4]{16} + \frac{1}{4(16)^{3/4}}(15.68-16)$$

$$\approx 2 + \frac{1}{4(8)}(-0.32)$$

$$\approx 2 - \frac{0.32}{32} = 2 - 0.01 = 1.99$$

$\sqrt[4]{15.68} \approx 1.99$

4. [10] Find the maximum rectangular area that can be fenced with \$4800 if two opposite sides of the rectangle will use fencing costing \$3 per foot and the remaining sides will use fencing costing \$8 per foot.



$A = \text{area} = xy$

$C = \text{cost of fencing} = 2x(3) + 2y(8) = 6x + 16y = 4800$; so

$6x \leq 4800 \to x \leq 800$; $3x + 8y = 2400 \to y = 300 - \frac{3}{8}x$

$A(x) = x(300 - \frac{3}{8}x)$, $0 \leq x \leq 800$

$A'(x) = 300 - \frac{3}{4}x$. $A'(x) = 0 \to 1200 - 3x = 0 \to x = 400$

$A(0) = 0 = A(800)$

$A(400) = 400(300 - 3(\frac{400}{8})) = 400(150) = 60000\ ft^2$

$=$ maximum area; $A''(x) = -\frac{3}{4} < 0$ for all x.

5. [10] Let $2x^2 + 3y^2 = 5$, where x and y both depend on the time variable t. At a certain time t, x is decreasing at a rate of $3cm/s$ while $x = 1$ and $y = -1$. Find how fast y is changing at that time. Is y increasing or decreasing?

At time t, $\frac{dx}{dt} = -3cm/s$

$\frac{d}{dt}(2x^2+3y^2) = \frac{d}{dt}(5) = 0 \rightarrow 2 \cdot 2x\frac{dx}{dt} + 3 \cdot 2y\frac{dy}{dt} = 0 \rightarrow 4x\frac{dx}{dt} + 6y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{4x}{6y}\frac{dx}{dt} = -\frac{2x}{3y}\frac{dx}{dt}$. $\left.\frac{dy}{dt}\right|_{x=1, y=-1} = -\frac{2(1)}{3(-1)}(-3) = -2cm/s$

Hence y is decreasing at a rate of $2cm/s$.

6. [20] Find the following indefinite integrals.

a) $\int (2x^{21} - \frac{9}{\sqrt[3]{x}} - \frac{4}{x^2})\, dx = 2\frac{x^{22}}{22} - 9\int x^{-1/3}dx - 4\int x^{-2}dx$

$= \frac{x^{22}}{11} - 9\frac{x^{2/3}}{2/3} - 4\left(\frac{x^{-1}}{-1}\right) + C$

$= \frac{x^{22}}{11} - \frac{27}{2}x^{2/3} + \frac{4}{x} + C$

b) $\int (2^x - \frac{3}{\sqrt{1-x^2}})\, dx = \frac{2^x}{\ln 2} - 3\arcsin x + C$

c) $\int (\cos x + \frac{2}{\sin^2 x})\, dx = \int (\cos x + 2\csc^2 x)dx$

$= \sin x - 2\cot x + C$

d) $\int \frac{\sqrt{x}}{1+x}\, dx = \int \frac{\sqrt{x}\cdot\sqrt{x}}{1+x}\cdot\frac{dx}{\sqrt{x}} = \int \frac{x}{1+x}\frac{dx}{\sqrt{x}}$

$u = \sqrt{x} \rightarrow x = u^2;\ dx = 2u\,du$

$= \int \frac{2u^2 du}{1+u^2} = 2\int \frac{u^2+1-1}{1+u^2}du$

$= 2\int (1 - \frac{1}{1+u^2})du = 2u - 2\arctan u +$

$= 2\sqrt{x} - 2\arctan(\sqrt{x}) + C$

7. [10] a) State the Mean Value Theorem. b) Show that the function f defined by $f(x) = \ln(x) - 2x$, x in $[1, 2]$, satisfies all the requirements of the Mean Value Theorem. c) Find all numbers x_0 in $(1, 2)$ such that $f'(x_0) = \frac{f(2)-f(1)}{2-1}$.

a) See Notes or text

b) f is continuous and differentiable on $(0, +\infty)$; so f is continuous on $[1,2]$ and differentiable on $(1,2)$; f satisfies all the requirements of the MVT.

c) $f'(x_0) = \frac{1}{x_0} - 2 = \frac{\ln(2) - 4 - (\ln(1) - 2)}{2-1} = \ln 2 - 4 + 2$, as $\ln(1) = 0$

$\frac{1}{x_0} \cancel{-2} = \ln 2 \cancel{-2}$, $\frac{1}{x_0} = \ln 2 \rightarrow x_0 = \frac{1}{\ln 2}$ in $(1,2)$.

Bonus question. [6] Evaluate the integral $\int \frac{\cos x + \sin x}{\cos^3 x}\, dx = \int \frac{\cos x}{\cos^3 x} + \frac{\sin x}{\cos x \cdot \cos^2 x}dx$

$= \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos x}\cdot\frac{1}{\cos^2 x}dx$

$= \int \sec^2 x + \tan x \sec^2 x\, dx$

$u = \tan x;\ du = \sec^2 x dx$

$= \tan x + \int u\, du$

$= \tan x + \frac{u^2}{2} + C$

$= \tan x + \frac{\tan^2 x}{2} + C$