MAC 2311 (Calculus I) — Answers
Test 2 , Wednesday October 07, 2015

Name: **PID:**

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. Guessing the correct answers won't earn you any credits. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University administration. Good Luck!

1. [30] Find the derivative of each of the following functions (Show all your work, and simplify your answers as much as possible; you will not get any credit(s) by guessing the correct answer(s).)

a) $f(x) = \frac{\sin x}{\sin x + \cos x}$

$$f'(x) = \frac{\cos x(\sin x + \cos x) - \sin x(\cos x - \sin x)}{(\sin x + \cos x)^2}$$
$$= \frac{\cancel{\cos x \sin x} + \cos^2 x - \cancel{\sin x \cos x} + \sin^2 x}{(\sin x + \cos x)^2}$$
$$= \frac{1}{(\sin x + \cos x)^2}$$

b) $g(x) = e^{2x}\tan^{-1}(2x)$

$$g'(x) = 2e^{2x}\tan^{-1}(2x) + e^{2x}\left(\frac{2}{1+4x^2}\right)$$
$$= 2e^{2x}\left(\tan^{-1}(2x) + \frac{1}{1+4x^2}\right)$$

c) $h(x) = 4^{x^3+5x-9} - 3\log_2(3x-8)$

$$h'(x) = (3x^2+5)(\ln 4)4^{x^3+5x-9} - \frac{9}{(3x-8)\ln 2}$$

d) $k(x) = -4x^{\frac{-1}{2}} - \frac{3}{\sqrt[7]{x^5}} + \frac{5}{x^4} + e^8$

$$k'(x) = 2x^{-3/2} + \frac{15}{7x\sqrt[7]{x^5}} - \frac{20}{x^5}$$

e) Use the logarithmic differentiation technique to find $\frac{dy}{dx}$ if $y = (e^x - \ln x)^{x^8}$.

$$\ln y = \ln\left[(e^x - \ln x)^{x^8}\right] = x^8 \ln(e^x - \ln x)$$
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(x^8 \ln(e^x - \ln x)\right) = 8x^7 \ln(e^x - \ln x) + \frac{x^8(e^x - \frac{1}{x})}{e^x - \ln x}$$
$$\frac{dy/dx}{y} = \uparrow \quad \text{so} \quad \frac{dy}{dx} = \left(8x^7 \ln(e^x - \ln x) + \frac{x^8(e^x - \frac{1}{x})}{e^x - \ln x}\right)(x - \ln x)^{x^8}$$

f) Use the implicit differentiation technique to find $\frac{dy}{dx}$ if $x^5y^3 + 4x^2 - 3y\cos(xy) = 4$.

$$\frac{d}{dx}\left(x^5y^3 + 4x^2 - 3y\cos(xy)\right) = \frac{d}{dx}(4) = 0$$
$$5x^4y^3 + 3x^5y^2\frac{dy}{dx} + 8x - 3\frac{dy}{dx}[\cos(xy)] - 3y\left[y + x\frac{dy}{dx}\right](-\sin(xy)) = 0$$
$$(3x^5y^2 - 3\cos(xy) + 3xy\sin(xy))\frac{dy}{dx} = -5x^4y^3 - 8x - 3y^2\sin(xy)$$
$$\frac{dy}{dx} = \frac{-(5x^4y^3 + 8x + 3y^2\sin(xy))}{3x^5y^2 - 3\cos(xy) + 3xy\sin(xy)}$$

2. [6] Find all values of x at which the line that is tangent to $y = \sec^2 x$ is parallel to the line $y - 4x = 5$.

We shall solve $\frac{dy}{dx} = 4$. Now $\frac{dy}{dx} = 2\sec^2 x\tan x = 2(\tan^2 x + 1)\tan x$

So solve $2\tan^3 x + 2\tan x = 4$ or $\tan^3 x + \tan x - 2 = 0$

Now $\tan^3 x + \tan x - 2 = (\tan x - 1)\underbrace{(\tan^2 x + \tan x + 2)}_{>0}$. So solve $\tan x = 1$

Hence $x = \frac{\pi}{4} + k\pi,\ k = 0, \pm 1, \pm 2, \ldots$

3. [6] Write down the correct statement.

a) Constant rule: $\frac{d}{dx}(k) = 0, \quad k = \text{constant}$

b) Power rule: $\frac{d}{dx}(x^r) = r x^{r-1}, \quad r = \text{real number}$

c) Product rule: $(fg)'(x) = f'(x)g(x) + g'(x)f(x)$

d) Quotient rule: $(f/g)'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

e) Chain rule: $(f(g))'(x) = g'(x) f'(g(x))$

f) Write down the definition: $f'(x_0) = \lim\limits_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$ or

$f'(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$

4. [10, **Bonus**] Use your knowledge of the derivative to evaluate each limit.

a) $\lim\limits_{x\to 1} \dfrac{\sin(\pi/x)}{x-1} = \dfrac{d}{dx}(\sin(\pi/x))\Big|_{x=1} = -\dfrac{\pi}{x^2}\cos(\pi/x)\Big|_{x=1} = -\pi \cos \pi = \pi$

b) $\lim\limits_{h\to 0} \dfrac{4\tan^{-1}(1+h) - \pi}{h} = \dfrac{d}{dx}(4\tan^{-1}x)\Big|_{x=1} = \dfrac{4}{1+x^2}\Big|_{x=1} = \dfrac{4}{1+1} = 2$

5. [10] Decide whether the statement is true or false. No explanation needed.

a) If $k(x) = \frac{g(x)}{x^3}$, then $k'(x) = \frac{g'(x)}{3x^2}$. False ; $k'(x) = (g'(x)x^3 - 3x^2 g(x))/x^6$

b) $\frac{d}{dx}(e^{h(x)}) = e^{h(x)}$. False ; $\frac{d}{dx}(e^{h(x)}) = h'(x)e^{h(x)}$

c) If $p(x) = q(x^2)$, then $p'(x) = q'(2x)$. False ; $p'(x) = 2x\, q'(x^2)$

d) If f is differentiable at 3, then f is continuous at 3. True, by DIC

e) If f is differentiable at $x = -5$, then $f'(-5)$ is the instantaneous rate of change of f with respect to x at $x = -5$. True

f) If $g(x) = m(x)x^3$, then $g'(x) = m'(x)x^3 + 3x^2m(x)$. True, product rule

g) If $m(x) = 9^7$, then $m'(x) = 7(9^6)$. False, by constant rule

h) If $k(x) = \csc^2(\ln(x^x)) - \cot^2(\ln(x^x))$, then $k'(x) = 0$. True, since $k(x) = 1$ for all x in D_k

i) If $g(x) = \ln(2x)$, then $g'(x) = \frac{1}{x}$. True, by chain rule

j) If $\lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h} = -5$, then $\lim\limits_{x\to 3} \dfrac{f(x) - f(3)}{x-3} = -5$. True, by definition of $f'(3)$.