MAC 2312 (Calculus II) — Key
Test 2, Monday November 25, 2013

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; no credits will be awarded to unexplained answers. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration.

1. [10] Use multiplication to find the first three nonzero terms of the Maclaurin series for $f(x) = e^{x^2}\sin x$.

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \dots; \quad e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots; \quad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$$e^{x^2}\sin x = \left(1 + x^2 + \frac{x^4}{2} + \dots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)$$

$$= x - \frac{x^3}{6} + x^3 + \frac{x^5}{120} - \frac{x^5}{6} + \frac{x^5}{2} + \dots$$

$$= x + \frac{5}{6}x^3 + \frac{41}{120}x^5 + \dots$$

2. [10] Find the Taylor polynomial of order four for $f(x) = \cos(x/3)$ about $x = \pi$.

$$f'(x) = -\frac{1}{3}\sin(x/3), \quad f''(x) = -\frac{1}{9}\cos(x/3), \quad f^{(3)}(x) = \frac{1}{27}\sin(x/3)$$

$$f^{(4)}(x) = \frac{1}{81}\cos(x/3), \quad \cos\frac{\pi}{3} = \frac{1}{2}, \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$P_4(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2}(x-\pi)^2 + \frac{f^{(3)}(\pi)}{6}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{24}(x-\pi)^4$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{6}(x-\pi) - \frac{1}{36}(x-\pi)^2 + \frac{\sqrt{3}}{54}\cdot\frac{1}{6}(x-\pi)^3 + \frac{1}{(162)(24)}(x-\pi)^4$$

3. [10] Decide whether each statement is true or false.

a) If $\lim_{k\to\infty} k^2 u_k = 1$, then the series $\sum u_k$ converges. True, by the limit comparison test.

b) If the series $\sum u_k$ converges, then $\lim_{k\to\infty} u_k = 0$. True, by the divergence test.

c) If $\lim_{k\to\infty} \sqrt[k]{|u_k|} = 1$, then the series $\sum |u_k|$ converges. False; set $u_k = (-1)^k\left(1+\frac{1}{k}\right)^k$, $k \geq 1$.

d) Every alternating series converges. False, pick $u_k = (-1)^k k$, $k \geq 1$.

e) If $0 < a_k \leq b_k$ for all $k \geq 1$, and $\sum a_k$ converges, then $\sum b_k$ converges too. False, pick $a_k = \frac{1}{k^2}$ and $b_k = \frac{1}{k}$

4. [10] a) Use a popular Maclaurin series to find the Maclaurin series for $f(x) = \frac{1}{1-x^2}$, and specify its interval of convergence. b) Find the derivative function f' of f, and use the Maclaurin series obtained in part a) and a well-known theorem to write down the Maclaurin series for f'.

a) $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, so $\frac{1}{1-x^2} = \sum_{k=0}^{\infty} (x^2)^k = \sum_{k=0}^{\infty} x^{2k}$, $I_C = (-1, 1)$

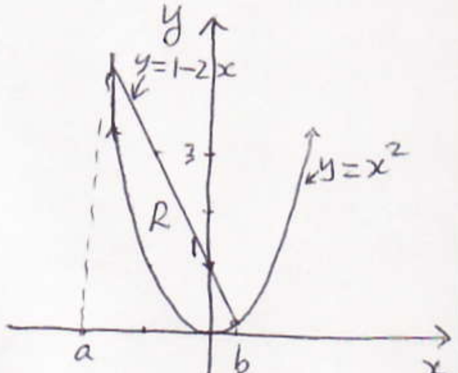
b) $f'(x) = \frac{-(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$

$$\frac{2x}{(1-x^2)^2} = \frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \frac{d}{dx}\sum_{k=0}^{\infty} x^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{d}{dx}(x^{2k}), \text{ by Term by term differentiation theorem}$$

$$= \sum_{k=1}^{\infty} 2k\, x^{2k-1}$$

5. [8] Sketch the region enclosed by the curves $y = x^2$, $2x + y = 1$, and find its area.



To find a and b, solve $x^2 = 1 - 2x$

or $x^2 + 2x - 1 = 0 \to \underbrace{x^2 + 2x + 1}_{(x+1)^2} - 2 = 0$

$(x+1)^2 - 2 = 0 \to (x+1)^2 = 2 \to x+1 = \pm\sqrt{2}$

so $a = -1-\sqrt{2}$, $b = -1+\sqrt{2}$

If A = area, then $A = \int_{-1-\sqrt{2}}^{-1+\sqrt{2}} 1 - 2x - x^2\, dx$

$$= x - x^2 - \frac{x^3}{3}\Big]_{1-\sqrt{2}}^{-1+\sqrt{2}} = \left(-1+\sqrt{2} - (-1+\sqrt{2})^2 - \frac{(-1+\sqrt{2})^3}{3}\right) - \left(-1-\sqrt{2} - (1+\sqrt{2})^2 + \frac{(1+\sqrt{2})^3}{3}\right)$$

6. [12] a) Find the exact length of the arc of the parametric curve $x = \cos(2t)$, $y = \sin(2t)$, $0 \leq t \leq \frac{\pi}{12}$.

$x'(t) = -2\sin(2t)$, $y'(t) = 2\cos(2t)$. $\sqrt{x'^2 + y'^2} = \sqrt{4(\sin^2(2t) + \cos^2(2t))} = \sqrt{4} = 2$

$$L = \int_0^{\pi/12} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{\pi/12} 2\, dt = 2t\Big]_0^{\pi/12} = 2\left(\frac{\pi}{12}\right) = \frac{\pi}{6}.$$

b) Find the volume of the solid that results when the region enclosed by the curves $y = 0$, $y = \frac{1}{\sqrt{1+x^2}}$, $x = 0$, and $x = 1$, is revolved about the x-axis.

$$V = \int_0^1 \pi\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx = \pi\int_0^1 \frac{dx}{1+x^2} = \pi \arctan x\Big]_0^1 = \pi\arctan 1 - \pi\arctan 0$$

$$= \pi\left(\frac{\pi}{4}\right) - 0$$

$$= \frac{\pi^2}{4}$$

7. [10] Determine the radius of convergence and the interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(-1)^k (x+3)^k}{k5^k}$.

Set $u_k = \frac{(-1)^k (x+3)^k}{k5^k}$.

$\ell = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{|x+3|^{k+1}}{(k+1)5^{k+1}} \cdot \frac{k5^k}{|x+3|^k} = \frac{|x+3|}{5} \lim_{k\to\infty} \frac{k}{k+1} = \frac{|x+3|}{5} < 1 \to |x+3| < 5$,

So $R = 5$. $-5 < x+3 < 5 \to -8 < x < 2$. At $x = -8$: $\sum_{k=1}^{\infty} \frac{(-1)^k(-8+3)^k}{k5^k} = \sum_{k=1}^{\infty} \frac{1}{k}$; diverges harmonic series. At $x = 2$: $\sum_{k=1}^{\infty} (-1)^k \frac{5^k}{k5^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$; converges by A.S.T.

Hence $I_c = (-8, 2]$

8. [6] Use the integral test to decide whether the infinite series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges or diverges.

$\int_2^{+\infty} \frac{dx}{x(\ln x)^2} = \lim_{r\to+\infty} \int_2^r \frac{dx}{x(\ln x)^2} = \lim_{r\to+\infty} \int_{\ln 2}^{\ln r} \frac{du}{u^2} = \lim_{r\to+\infty} -\frac{1}{u}\Big]_{\ln 2}^{\ln r} = -\lim_{r\to+\infty} \frac{1}{\ln r} + \frac{1}{\ln 2}$

$= 0 + \frac{1}{\ln 2}$

$u = \ln x$

$du = \frac{dx}{x}$

The improper integral converges; so $= \frac{1}{\ln 2}$.

the series converges by the integral test.

9. [8] Show that the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ satisfies the requirements of the alternating series test. Find a value of n for which the n^{th} partial sum is ensured to approximate the series to within three-decimal place accuracy.

Set $a_k = \frac{1}{k^3}$; then $a_{k+1} = \frac{1}{(k+1)^3} < \frac{1}{k^3}$ for all $k \geq 1$; so (a_k) is decreasing. Further $\lim_{k\to+\infty} a_k = \lim_{k\to+\infty} \frac{1}{k^3} = 0$; so series satisfies the requirements of the A.S.T. If $S_n = \sum_{k=1}^{n} \frac{(-1)^k}{k^3}$, and $S = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$, then $|S_n - S| \leq \frac{1}{(n+1)^3}$. For $|S_n - S| \leq 0.0005$, it is enough that $\frac{1}{(n+1)^3} \leq 5\times10^{-4}$ or $\frac{10000}{5} \leq (n+1)^3 \to 10\sqrt[3]{2} \leq n+1 \to 10\sqrt[3]{2} - 1 \leq n$.

10. [8] Use the comparison test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{3+k2^k}$ converges or diverges.

$\frac{1}{3+k2^k} < \frac{1}{k2^k} \leq \frac{1}{2^k}$ for all $k \geq 1$. Now $\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ converges as a geometric series with ratio $r = \frac{1}{2} < 1$; so $\sum_{k=1}^{\infty} \frac{1}{3+k2^k}$ converges by the comparison test, since the bigger series $\sum \frac{1}{2^k}$ converges.

11. [8] Write the nth partial sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^2+6k+8}$ as a telescoping sum, and use it to show that the given series converges, and find its sum.

$S_n = \sum_{k=1}^{n} \frac{1}{k^2+6k+8} = \sum_{k=1}^{n} \frac{1}{(k+2)(k+4)} = \sum_{k=1}^{n} \left(\frac{1/2}{k+2} - \frac{1/2}{k+4}\right) = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k+2} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k+4}$

Now $\sum_{k=1}^{n} \frac{1}{k+4} = \sum_{k=3}^{n+2} \frac{1}{k+2} = \sum_{k=3}^{n} \frac{1}{k+2} + \frac{1}{n+3} + \frac{1}{n+4}$; hence

$S_n = \frac{1}{2}\left(\frac{1}{1+2} + \frac{1}{2+2}\right) + \cancel{\frac{1}{2}\sum_{k=3}^{n} \frac{1}{k+2}} - \cancel{\frac{1}{2}\sum_{k=3}^{n} \frac{1}{k+2}} - \frac{1}{2}\left(\frac{1}{n+3} + \frac{1}{n+4}\right)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{7}{24} - \frac{1}{2}\left(\frac{1}{n+3} + \frac{1}{n+4}\right)\right) = \frac{7}{24} - 0 = \frac{7}{24}$; so series converges with sum $\frac{7}{24}$.