MAC 2313 (Calculus III) — Key
Test 2, Monday November 25, 2013

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. *You will not get any credit to any of the problems if you do not show your work.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Total=103 points.**

1. [10] Use Lagrange multipliers to find the minimum and maximum values of $f(x,y) = 8x - 2y + 1$ subject to the constraint $4x^2 + y^2 = 5$. Also state where these extreme values occur.

Set $h(x,y) = 8x - 2y + 1 - \lambda(4x^2+y^2-5)$

$\nabla h(x,y) = \langle 8 - 8\lambda x, -2 - 2\lambda y\rangle = \vec{0} \to x = 1/\lambda$ and $y = -\frac{1}{\lambda}$

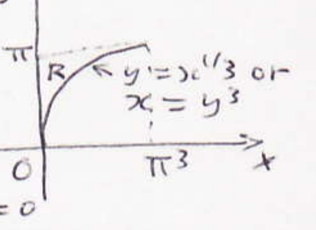
$4x^2+y^2 = 5 \to 4\left(\frac{1}{\lambda^2}\right) + \frac{1}{\lambda^2} = 5$ or $\frac{5}{\lambda^2} = 5 \to 5 = 5\lambda^2 \to \lambda^2 = 1$

So $\lambda = \pm 1$.

i) $\lambda = 1 \to (1,-1) \to f(1,-1) = 8 - 2(-1) + 1 = 11$ = maximum value

ii) $\lambda = -1 \to (-1,1) \to f(-1,1) = 8(-1) - 2(1) + 1 = -9$ = minimum value

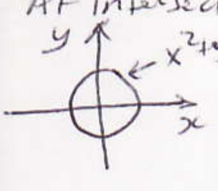
2. [10] Evaluate the integral $\int\int_{\mathcal{R}} \cos(y^4)\,dA$, where $\mathcal{R}$ is the region in the first quadrant enclosed by the curves $x = 0$, $y = x^{\frac{1}{3}}$, and $y = \pi$.



$$\iint_R \cos(y^4)\,dA = \int_0^{\pi^3}\int_{x^{1/3}}^{\pi} \cos(y^4)\,dy\,dx$$
$$= \int_0^{\pi}\int_0^{y^3} \cos(y^4)\,dx\,dy$$
$$= \int_0^{\pi} y^3\cos(y^4)\,dy = \frac{\sin(y^4)}{4}\Big]_0^{\pi} = \frac{\sin(\pi^4)}{4}, \text{ as } \sin 0 = 0$$

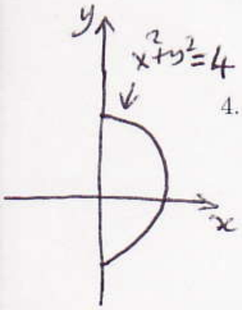
3. [15] a) Use cylindrical coordinates to evaluate the volume of the solid bounded above by the parabolic cylinder $z = 2 - y^2$ and below by the paraboloid $z = 2x^2 + y^2$. b) Write down the coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the centroid of that solid as iterated integrals including all limits of integration, but do not evaluate these integrals.

At intersection: $2 - y^2 = 2x^2 + y^2$ or $2 = 2x^2 + 2y^2$ or $x^2+y^2 = 1$.



a) $$V = \int_0^{2\pi}\int_0^1\int_{2r^2\cos^2\theta + r^2\sin^2\theta}^{2 - r^2\sin^2\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\left(2 - 2r^2(\cos^2\theta + \sin^2\theta)\right)dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 2r - 2r^3\,dr\,d\theta = 2\pi\left[r^2 - \frac{r^4}{2}\right]_0^1 = 2\pi\left(\frac{1}{2}\right) = \pi$$

b) $$\bar{x} = \frac{1}{\pi}\int_0^{2\pi}\int_0^1\int_{2r^2\cos^2\theta + r^2\sin^2\theta}^{2 - r^2\sin^2\theta} r\cos\theta \cdot r\,dz\,dr\,d\theta = 0 \text{ by symmetry; similarly}$$
$$\bar{y} = 0.$$
$$\bar{z} = \frac{1}{\pi}\int_0^{2\pi}\int_0^1\int_{2r^2\cos^2\theta + r^2\sin^2\theta}^{2 - r^2\sin^2\theta} z r\,dz\,dr\,d\theta$$



4. [7] Express the triple integral $J = \displaystyle\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} \frac{1}{x^2+y^2+z^2+1}\,dz\,dy\,dx$ using spherical coordinates, but do not evaluate the resulting integral.

$$J = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^2 \frac{\rho^2\sin\phi}{\rho^2+1}\,d\rho\,d\phi\,d\theta$$

$\sqrt{x^2+y^2} \le z \le \sqrt{4-x^2-y^2} \to z^2 \le 4 - x^2 - y^2 \to x^2+y^2+z^2 \le 4$

$\rho\sin\phi \le \rho\cos\phi \to \tan\phi \le 1 \to 0 \le \phi \le \frac{\pi}{4}$

$\rho^2 \le 4 \to 0 \le \rho \le 2$

5. [14] a) Find an equation for the tangent plane to the parametric surface $\vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + u\,\vec{k}$ at the point given by $u = 1$ and $v = \pi$.

a) $\vec{n} = \vec{r}_u \times \vec{r}_v(1,\pi)$ = a normal to tangent plane

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (0 - u\cos v)\vec{i} - (0 + u\sin v)\vec{j} + (u\cos^2 v + u\sin^2 v)\vec{k}$$

$\vec{r}_u \times \vec{r}_v(1,\pi) = -\cos\pi\,\vec{i} - \sin\pi\,\vec{j} + \vec{k} = \vec{i} + \vec{k}$; Point $= (-1, 0, 1)$ when $u = 1$ & $v = \pi$

Eqn: $x + 1 + z - 1 = 0$ or $x + z = 0$.

b) Evaluate the mass of the solid enclosed by the parametric surface in part a) if $1 \le u \le 3$ and $0 \le v \le 2\pi$, and its density is $\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$.

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2u^2}$

$$M = \int_1^3 \int_0^{2\pi} \delta \|\vec{r}_u \times \vec{r}_v\| \, dv\, du = \int_1^3 \int_0^{2\pi} \sqrt{2u^2}\cdot\sqrt{2u^2}\, dv\, du = 2\pi \int_1^3 2u^2\, du$$
$$= 4\pi \left[\frac{u^3}{3}\right]_1^3 = \frac{4\pi}{3}(26) = \frac{104\pi}{3}$$

6. [10] Let $G = \{(x,y,z); 1 - e^x \le y \le 3 - e^x,\ 1 - y \le 2z \le 2 - y,\ y \le e^x \le y + 4\}$ be a solid in 3-space. Set $u = y + e^x$, $v = y + 2z$, $w = e^x - y$.

a) Find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ and express it in terms of u, v, and w.

b) Express the volume of G in the uvw-coordinates system including all integration limits, but don't evaluate it.

a) $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} e^x & 1 & 0 \\ 0 & 1 & 2 \\ e^x & -1 & 0 \end{vmatrix} = (0+2)e^x - 1(0 - 2e^x) = 4e^x$$

Now $w + u = 2e^x$; so $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 2(2e^x) = 2(u+w)$, and

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{2(u+w)}$$

b) $1 \le u \le 3$, $1 \le v \le 2$, $0 \le w \le 4$

$$V = \int_1^3 \int_1^2 \int_0^4 \frac{1}{2(u+w)}\, dw\, dv\, du$$

7. [10] a) Find $\operatorname{div}\mathbf{F}(x,y,z)$ and $\operatorname{curl}\mathbf{F}(x,y,z)$ if $\mathbf{F}(x,y,z) = z^2x\,\vec{i} + x^2y\,\vec{j} + y^2z\,\vec{k}$.

b) Find the Laplacian of $g(x,y,z) = x^2y^2z^2$.

a) $\operatorname{div} F(x,y,z) = \partial_x(z^2x) + \partial_y(x^2y) + \partial_z(y^2z) = z^2 + x^2 + y^2$

$$\operatorname{Curl} F(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ z^2x & x^2y & y^2z \end{vmatrix} = (2yz - 0)\vec{i} - (0 - 2zx)\vec{j} + (2xy - 0)\vec{k}$$
$$= 2yz\,\vec{i} + 2zx\,\vec{j} + 2xy\,\vec{k}$$

b) $$\nabla^2 g(x,y,z) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 2y^2z^2 + 2x^2z^2 + 2x^2y^2$$

$\frac{\partial g}{\partial x}(x,y,z) = 2xy^2z^2$

$\frac{\partial^2 g}{\partial x^2}(x,y,z) = 2y^2z^2$; by symmetry $\frac{\partial^2 g}{\partial y^2} = 2x^2z^2$, $\frac{\partial^2 g}{\partial z^2} = 2x^2y^2$

8. [15] Let $F(x,y) = (3x^2y+2x)\vec{i} + (x^3+y)\vec{j}$. a) Show that F is conservative. b) Find a potential function φ for F. c) Evaluate the line integral $\int_C (3x^2y+2x)\,dx + (x^3+y)\,dy$ along the curve C parametrized by $\vec{r}(t) = \ln(1+t)\vec{i} + \sin^{-1} t\,\vec{j}$, $0 \le t \le 1$.

a) $\partial_x(x^3+y) = 3x^2 = \partial_y(3x^2y+2x)$; so F is conservative.

b) Thanks to a), there exists a potential function φ with

(i) $\varphi_x(x,y) = 3x^2y+2x$, (ii) $\varphi_y(x,y) = x^3+y$

Integrate (i) w.r.t. x:

$\varphi(x,y) = \int(3x^2y+2x)\,dx = x^3y + x^2 + k(y)$ (iii)

Differentiate (iii) w.r.t. y:

$\varphi_y(x,y) = x^3 + k'(y)$

$= x^3 + y$, by (ii)

So $k'(y) = y$, and we may choose $k(y) = \frac{y^2}{2}$

Hence $\varphi(x,y) = x^3y + x^2 + \frac{y^2}{2}$

c) $\int_C (3x^2y+2x)\,dx + (x^3+y)\,dy = \int_C F\cdot d\vec{r} = \varphi(\ln(1+1), \sin^{-1}(1)) - \varphi(\ln(1+0), \sin^{-1}0)$

$= \varphi(\ln 2, \frac{\pi}{2}) - \varphi(0,0)$

$= (\ln 2)^3\frac{\pi}{2} + (\ln 2)^2 + \frac{\pi^2}{8}$, as $\varphi(0,0) = 0$

9. [12] a) State Green's Theorem.

See text or notes

b) Use Green's Theorem to evaluate the line integral $K = \oint_C (2\arctan(y/(x+1)) + e^{\cos x} - 2y)\,dx + \ln((x+1)^2+y^2)\,dy$, where C is the boundary of the region in the first quadrant enclosed by the curves $y = 3x$ and $y = x^3$ with a counterclockwise orientation.

By Green's Theorem,

$$K = \iint_R \partial_x\left(\ln((x+1)^2+y^2)\right) - \partial_y\left(2\tan^{-1}(y/(x+1)) + e^{\cos x} - 2y\right)dA$$

$$= \int_0^{\sqrt{3}}\int_{x^3}^{3x}\left(\cancel{\frac{2(x+1)}{(x+1)^2+y^2}} - \frac{2}{x+1}\cdot\cancel{\frac{1}{1+\frac{y^2}{(x+1)^2}}} + 2\right)dy\,dx$$

$$= 2\int_0^{\sqrt{3}} 3x - x^3\,dx$$

$$= \left[3x^2 - \frac{2x^4}{4}\right]_0^{\sqrt{3}} = 3(3) - \frac{9}{2} = \frac{9}{2}.$$

$3x = x^3$

$x(3-x^2) = 0$

$x = 0$ or $x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$