MAC 2313 (Calculus III) — Answers
Test 2, Wednesday October 07, 2015

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve any of the points assigned to any question. *You will not get any credit if you do not show the steps to your answers.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. 2 pages. Total=65 points. Good luck.**

1 . [20] a) Find an equation for the tangent plane and parametric equations for the normal line to the hyperboloid $2z^2 - y^2 - 4x^2 = 4$ at the point $(1,-2,\sqrt{6})$.

Set $F(x,y,z) = 2z^2 - y^2 - 4x^2 - 4$. $\nabla F(x,y,z) = \langle -8x, -2y, 4z\rangle$

$\nabla F(1,-2,\sqrt{6}) = \langle -8, 4, 4\sqrt{6}\rangle$; So $\vec{n} = \langle -2, 1, \sqrt{6}\rangle$ is a normal to plane

Equation of tangent plane: $-2(x-1) + (y+2) + \sqrt{6}(z-\sqrt{6}) = 0$.

parametric equations of normal line

$x = 1 - 2t$, $y = -2 + t$, $z = \sqrt{6} + \sqrt{6}t$

b) Let $f(x,y) = x^2 - 6xy + 3y^2 - 6y + 2$. Find all the critical points of f and classify each of them as a local maximum, a local minimum, or a saddle point.

$f_x(x,y) = 2x - 6y$, $f_y(x,y) = -6x + 6y - 6$

$f_x(x,y) = 0 \to x - 3y = 0 \to x = 3y$. $f_y(x,y) = 0 \to -x + y - 1 = 0 \to x = y - 1$

So $3y = y - 1 \to 2y = -1 \to y = -1/2$; so $x = -3/2$

C.P.: $(-3/2, -1/2)$. $f_{xx}(x,y) = 2$, $f_{xy}(x,y) = -6$, $f_{yy}(x,y) = 6$

$\Delta = f_{xy}(-3/2,-1/2)^2 - f_{xx}(-3/2,-1/2) f_{yy}(-3/2,-1/2) = 36 - 12 = 24 > 0$;

So f has a saddle point at $(-3/2, -1/2)$

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2. [10] Let $f(x,y) = 2x^2 - 3y^2$. Show that f differentiable at the point $(2,1)$?

$f_x(x,y) = 4x$, $f_y(x,y) = -6y$

$$\lim_{(h,k)\to(0,0)} \frac{f(2+h,1+k) - f(2,1) - h f_x(2,1) - k f_y(2,1)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{2(2+h)^2 - 3(1+k)^2 - 5 - 8h + 6k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{2(4+4h+h^2) - 3(1+2k+k^2) - 5 - 8h + 6k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{\cancel{8} + \cancel{8h} + 2h^2 - \cancel{8} - \cancel{6k} - 3k^2 + \cancel{6k} - \cancel{8h}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{2h^2 - 3k^2}{\sqrt{h^2+k^2}}$$

Set $h = r\cos\theta$, $k = r\sin\theta$, $r > 0$, $0 \le \theta < 2\pi$;

$$= \lim_{r\to 0^+} \frac{r^2(2\cos^2\theta - 3\sin^2\theta)}{r}$$

$$= \lim_{r\to 0^+} r(2\cos^2\theta - 3\sin^2\theta)$$

$= 0$, for all θ.

Hence f is differentiable at $(2,1)$.

3. [12] Set $u = xe^y$, $v = ye^x$, and $z(u,v) = \sin(xy)$. First, use implicit partial differentiation to find the partial derivatives x_u and y_u, then, use the chain rule to find the partial derivative z_u.

$\frac{\partial}{\partial u}(u) = \frac{\partial}{\partial u}(xe^y)$ $\qquad$ $\frac{\partial}{\partial u}(v) = \frac{\partial}{\partial u}(ye^x)$

$1 = (x_u + y_u x)e^y$ $\qquad$ $0 = (y_u + x_u y)e^x \to y x_u + y_u = 0 \to y_u = -y x_u$

so $1 = (x_u - yx x_u)e^y$; hence $x_u = \frac{e^{-y}}{1-xy}$ and $y_u = \frac{-ye^{-y}}{1-xy}$

$$\begin{aligned} z_u &= \partial_x(\sin(xy))\, x_u + \partial_y \sin(xy)\, y_u \\ &= y\cos(xy)\, x_u + x\cos(xy)\, y_u \\ &= \frac{(y - xy)\cos(xy)\cdot e^{-y}}{1-xy} \end{aligned}$$

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4. [9] Let $f(x,y,z) = x^2y + y^2z + z^2x$. a) Find the gradient of f at $P(1,1,-1)$. b) Find a unit vector in the direction in which f increases most rapidly at the point $P(1,1,-1)$, and find the rate of change of f at P in that direction. c) Find the directional derivative of f in the direction of the vector $\vec{w} = \vec{i} - 3\vec{j} + \vec{k}$ at the point $A(1,2,1)$

a) $\nabla f(x,y,z) = \langle 2xy + z^2, 2yz + x^2, y^2 + 2zx \rangle$

$\nabla f(P) = \langle 3, -1, -1 \rangle$

b) $\vec{u} = \frac{\nabla f(P)}{\|\nabla f(P)\|} = \frac{\langle 3, -1, -1\rangle}{\sqrt{11}}$, rate of change $= \|\nabla f(P)\| = \sqrt{11}$

c) $D_{\vec{w}} f(A) = \nabla f(A) \cdot \frac{\vec{w}}{\|\vec{w}\|} = \langle 5, 5, 6\rangle \cdot \frac{\langle 1, -3, 1\rangle}{\sqrt{11}} = \frac{5 - 15 + 6}{\sqrt{11}} = \frac{-4}{\sqrt{11}}$

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5. [8] Let $h(x,y,z) = xye^{yz}$. Find the local linear approximation for h about the point $P(1,0,1)$, and use it to approximate $h(1.01, -0.01, 0.98)$.

$h_x(x,y,z) = ye^{yz}$, $h_y(x,y,z) = (x + xyz)e^{yz}$, $h_z(x,y,z) = xy^2e^{yz}$

$$\begin{aligned} L(x,y,z) &= h(P) + h_x(P)(x-1) + h_y(P)y + h_z(P)(z-1) \\ &= 0 + 0(x-1) + y + 0(z-1) \\ &= y \end{aligned}$$

$h(1.01, -0.01, 0.98) \approx L(1.01, -0.01, 0.98) = -0.01$

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6. [6] Let $f(x,y,z) = e^{x^2+2x+y^2+z^2-3}$. Describe precisely the level surface of constant k for f when $k = 1$, and when $k = -1$.

For $k=1$: $e^{x^2+2x+y^2+z^2-3} = 1$; so $x^2+2x+y^2+z^2-3 = 0$ or $(x+1)^2 + y^2 + z^2 = 4$; sphere centered at $(-1,0,0)$ with radius 2.

For $k=-1$: $e^{x^2+2x+y^2+z^2-3} = -1$; no level surface, since $e^u > 0$ for all real number u.