MAC 2313 (Calculus III) — Answers
Test 2, Thursday June 15, 2017

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work. Guessing the correct answers won't give you any credits. You must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may use the back of the page to show your work. 3 pages. Total=90 points. Always do your best.

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1 . [15] a) Set $h(s) = f(2+3s, 3-4s, 5+6s)$, where the function $f = f(x,y,z)$ is differentiable. Use the chain rule to find the derivative: $\frac{dh}{ds}(s) = \frac{d}{ds}(2+3s) f_x(2+3s, 3-4s, 5+6s) + \frac{d}{ds}(3-4s) f_y(2+3s, 3-4s, 5+6s)$

$$+ \frac{d}{ds}(5+6s) f_z(2+3s, 3-4s, 5+6s)$$

$$= 3 f_x(2+3s, 3-4s, 5+6s) - 4 f_y(2+3s, 3-4s, 5+6s) + 6 f_z(2+3s, 3-4s, 5+6s)$$

b) Find an equation for, and identify the surface that results when the surface $3x^2 - 4y^2 - z^2 = 1$ is ~~revolved~~ reflected about the plane i) $x = z$, ii) $y = 0$.

i) switch x and z: $3z^2 - 4y^2 - x^2 = 1$; hyperboloid of two sheets along z-axis

ii) change y to $-y$: $3x^2 - 4(-y)^2 - z^2 = 1$
$3x^2 - 4y^2 - z^2 = 1$; hyperboloid of two sheets along x-axis

c) Let $w(x,y,z) = \ln(1+xyz)$. Find the gradient of w at the point $B(1,-1,-2)$.

$$\nabla w(x,y,z) = \left\langle \frac{yz}{1+xyz}, \frac{xz}{1+xyz}, \frac{xy}{1+xyz} \right\rangle$$

$$\nabla w(B) = \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{-1}{3} \right\rangle$$

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2. [15] a) Write down the definition of "f is differentiable at (x_0, y_0)". b) Use the definition in a) to show that the function f given by $f(x,y) = x^2 - 3y^2$ is differentiable at the point $(-2, 1)$.

a) $f(x_0,y_0)$, $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ are all defined and

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - h f_x(x_0,y_0) - k f_y(x_0,y_0)}{\sqrt{h^2+k^2}} = 0$$

b) $f(-2,1) = 4 - 3 = 1$, $f_x(x,y) = 2x$, $f_y(x,y) = -6y$, $f_x(-2,1) = -4$
$f_y(-2,1) = -6$

$$\lim_{(h,k)\to(0,0)} \frac{(-2+h)^2 - 3(1+k)^2 - 1 - h(-4) - k(-6)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\cancel{4} - \cancel{4h} + h^2 - 3(\cancel{1} + \cancel{2k} + k^2) - \cancel{1} + \cancel{4h} + \cancel{6k}}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2 - 3k^2}{\sqrt{h^2+k^2}}, \quad \text{set } h = r\cos\theta,\ k = r\sin\theta,\ r > 0,\ 0 \le \theta \le 2\pi$$

$$= \lim_{r\to 0^+} \frac{r^2(\cos^2\theta - 3\sin^2\theta)}{r} = \lim_{r\to 0^+} r(\cos^2\theta - 3\sin^2\theta) = 0, \text{ for all } \theta;$$

So f is differentiable at the point $(-2, 1)$.

3. [10] a) Find an equation for the level surface of the function g defined by $g(x,y,z) = e^{1+xyz}$ that passes through the point $D(1,-2,-1)$. Simplify the equation as much as possible.

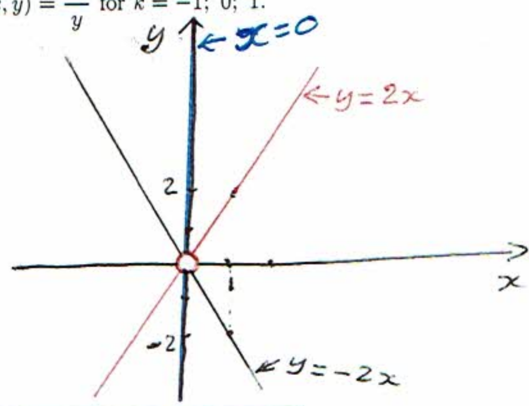
$e^{1+xyz} = e^{1+1(-2)(-1)} = e^{1+2}$; so $1+xyz = 1+2$, as the exponential function is one-to-one; hence $xyz = 2$ is the required level surface

b) Find an equation for, and sketch the level curve of constant k of the function $\ell(x,y) = \dfrac{2x}{y}$ for $k = -1;\ 0;\ 1.$

$k = -1$: $\dfrac{2x}{y} = -1 \to 2x = -y$ or $y = -2x$

$k = 0$: $\dfrac{2x}{y} = 0 \to 2x = 0 \to x = 0$

$k = 1$: $\dfrac{2x}{y} = 1 \to 2x = y$ or $y = 2x$



4. [7] Evaluate each limit. If a limit does not exist, explain why.

a) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{\sin(x^4-y^4)}{x^2-y^2}$

$$= \lim_{(x,y)\to(1,-1)} \frac{\sin(x^4-y^4)}{x^4-y^4}\cdot(x^2+y^2)$$

$$= \lim_{u\to 0}\frac{\sin u}{u}\cdot \lim_{(x,y)\to(1,-1)}(x^2+y^2)$$

$$= (1)(1+1) = 2$$

b) $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{xyz}{x^2+y^2+z^2} = \frac{1(1)(1)}{1+1+1} = \frac{1}{3}$

5. [13] Let $f(x,y,z) = yz^3\cos(xy)$. a) Find $f(-2y, 3z, x^2)$. b) Find the partial derivatives f_x, f_y and f_z. c) Find the local linear approximation L to f at the point $A(0,1,-1)$. d) Use the local linear approximation L to f at A to approximate $f(0.03, 0.99, -1.02)$.

a) $f(-2y, 3z, x^2) = 3z(x^2)^3\cos(-2y(3z)) = 3zx^6\cos(-6yz) = 3zx^6\cos(6yz)$

b) $f_x(x,y,z) = -y^2z^3\sin(xy)$, $f_y(x,y,z) = z^3\cos(xy) - xyz^3\sin(xy)$

$f_z(x,y,z) = 3yz^2\cos(xy)$

c) $L(x,y,z) = f(A) + f_x(A)(x-0) + f_y(A)(y-1) + f_z(A)(z+1)$

$= -1 + 0x + (-1-0)(y-1) + 3(z+1)$

$= -1 - (y-1) + 3(z+1)$

d) $f(0.03, 0.99, -1.02) \approx L(0.03, 0.99, -1.02)$

$\approx -1 - (0.99-1) + 3(-1.02+1)$

$\approx -1 - (-0.01) + 3(-0.02)$

$\approx -1 + 0.01 - 0.06$

≈ -1.05

6. [10] Use implicit differentiation to find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$ if $x^3 + z^2 y - e^{xy} = -9$.

$\frac{\partial}{\partial x}(x^3 + z^2 y - e^{xy}) = \frac{\partial}{\partial x}(-9) = 0$; $3x^2 + z^2\frac{\partial y}{\partial x} - (y + x\frac{\partial y}{\partial x})e^{xy} = 0$

$(z^2 - xe^{xy})\frac{\partial y}{\partial x} = -3x^2 + ye^{xy}$; so $\frac{\partial y}{\partial x} = \frac{ye^{xy} - 3x^2}{z^2 - xe^{xy}}$

$\frac{\partial}{\partial z}(x^3 + z^2 y - e^{xy}) = \frac{\partial}{\partial z}(-9) = 0$; $2zy + z^2\frac{\partial y}{\partial z} - x(\frac{\partial y}{\partial z})e^{xy} = 0$

$(z^2 - xe^{xy})\frac{\partial y}{\partial z} = -2zy$; so $\frac{\partial y}{\partial z} = \frac{-2zy}{z^2 - xe^{xy}}$

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7. [20] a) Find an equation for the tangent plane and parametric equations for the normal line to the surface $5x^3 + 2y^2 - 3z^2 = 10$ at the point $P(1, 2, -1)$.

Set $f(x,y,z) = 5x^3 + 2y^2 - 3z^2 - 10$. $\nabla f(x,y,z) = \langle 15x, 4y, -6z \rangle$

$\nabla f(P) = \langle 15, 8, 6 \rangle$ = a normal to tangent plane = a vector // normal line

Equation for tangent plane: $15(x-1) + 8(y-2) + 6(z+1) = 0$

parametric equations for normal line: $x = 1 + 15t$

$y = 2 + 8t$

$z = -1 + 6t, \quad -\infty < t < \infty.$

b) Find parametric equations for the tangent line to the curve of intersection of the plane $4x + 2y + z = 6$ and the paraboloid $x^2 + y^2 - z = 5$ at the point $Q(-2, 3, 8)$.

Set $f(x,y,z) = 4x + 2y + z - 6$, $g(x,y,z) = x^2 + y^2 - z - 5$

$\nabla f(x,y,z) = \langle 4, 2, 1 \rangle$, $\nabla g(x,y,z) = \langle 2x, 2y, -1 \rangle$

$\nabla f(Q) \times \nabla g(Q)$ = a vector parallel to tangent line

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & 1 \\ -4 & 6 & -1 \end{vmatrix} = (-2-6)\vec{i} - (-4+4)\vec{j} + (24+8)\vec{k}$$

$$= -8\vec{i} + 32\vec{k}$$

$$= -8(\vec{i} - 4\vec{k})$$

Parametric equations

$x = -2 + t$

$y = 3$

$z = 8 - 4t, \quad -\infty < t < \infty.$