MAC 2313 (Calculus III) — Key
Test 2, Wednesday April 09, 2014

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question; answers without any explanation won't get any credits. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good luck.

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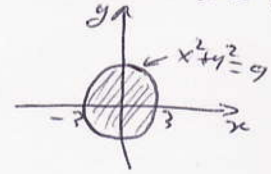
1. [25]

a) Let G be the solid bounded below by the plane $z = 1$ and above by the cone $z = 4 - \sqrt{x^2 + y^2}$. Express the volume V of G (Do not evaluate any of the integrals involved, but include all integration limits) using:

a) rectangular coordinates: $V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{4-\sqrt{x^2+y^2}} dz\,dy\,dx$

At intersection $1 = 4 - \sqrt{x^2+y^2}$
so $\sqrt{x^2+y^2} = 3$
or $x^2+y^2 = 9$



b) cylindrical coordinates: $V = \int_{0}^{2\pi} \int_{0}^{3} \int_{1}^{4-r} r\,dz\,dr\,d\theta$

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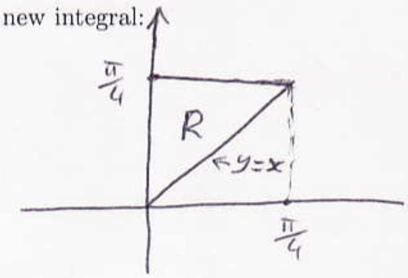
c) Reverse the order of integration in the following integral, but do not evaluate the new integral:

$$\int_0^{\frac{\pi}{4}} \int_x^{\frac{\pi}{4}} \frac{\tan(y)}{y}\,dy\,dx = \int_0^{\frac{\pi}{4}} \int_0^{y} \frac{\tan(y)}{y}\,dx\,dy$$

R is viewed as type I region

R is viewed as type II region



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d) Let $\mathbf{F}(x, y, z) = (x^2 - 2yz)\vec{i} + (y^2 - 2xz)\vec{j} + (z^2 - xy)\vec{k}$. Find div$\mathbf{F}$ and curl$\mathbf{F}$.

$\operatorname{div} F(x,y,z) = \partial_x(x^2-2yz) + \partial_y(y^2-2xz) + \partial_z(z^2-xy) = 2x + 2y + 2z$

$$\operatorname{curl} F(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2-2yz & y^2-2xz & z^2-xy \end{vmatrix} = (\partial_y(z^2-xy) - \partial_z(y^2-2xz))\vec{i} - (\partial_x(z^2-xy) - \partial_z(x^2-2yz))\vec{j} + (\partial_x(y^2-2xz) - \partial_y(x^2-2yz))\vec{k}$$

$$= (-x+2x)\vec{i} - (-y+2y)\vec{j} + (-2z+2z)\vec{k} = x\vec{i} - y\vec{j}$$

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2. [10] Use the change of variables $u = x + y$, $v = x - y$ to evaluate $\iint_R (x - y)\cos(x^2 - y^2)\,dA$, where R is the rectangular region enclosed by the lines $x + y = 0$, $x + y = 1$, $x - y = 1$, $x - y = 3$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

$$\iint_R (x-y)\cos(x^2-y^2)\,dA = \int_1^3 \int_0^1 v\cos(uv)\cdot\frac{1}{2}\,du\,dv = \frac{1}{2}\int_1^3 \Big[\sin(uv)\Big]_0^1 dv = \frac{1}{2}\int_1^3 \sin v\,dv = \frac{1}{2}\Big[-\cos v\Big]_1^3$$

$$= \frac{1}{2}(-\cos 3 + \cos 1)$$

3. [20] a) State Green's Theorem, including all hypotheses and the conclusion. See text or notes.

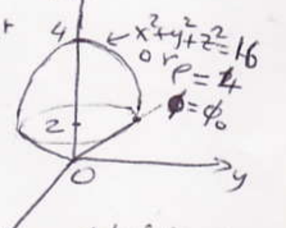
b) Use spherical coordinates to find the volume and the centroid of the solid bounded below by the plane $z = 2$ and above by the sphere $x^2 + y^2 + z^2 = 16$. Just set up the triple integral for the volume and each coordinate of the centroid including all integration limits, but do not evaluate any of those integrals.

$2 \le z \to 2 \le \rho\cos\phi \to \frac{2}{\cos\phi} \le \rho$ or $2\sec\phi \le \rho$

$x^2+y^2+z^2 = 16 \to \rho^2 = 16 \to \rho = 4$; so $2\sec\phi \le \rho \le 4$

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_{2\sec\phi}^{4} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

To find ϕ_0, notice that $\rho\cos\phi_0 = 2$, $\rho\sin\phi_0 = \sqrt{12} = 2\sqrt{3}$. So $\tan\phi_0 = \sqrt{3}$, $\phi_0 = \frac{\pi}{3}$

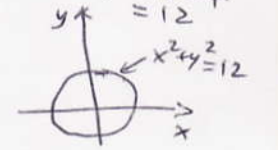


$\bar{x} = 0,\ \bar{y} = 0$ since solid is symmetric about z-axis

$$\bar{z} = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/3}\int_{2\sec\phi}^{4} \rho\cos\phi \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Solid is contained in the cone as shown

At intersection $x^2+y^2 = 16-4 = 12$



As an exercise, you may now evaluate those integrals to prepare for the final exam.

4. [13] Consider the parametric surface given by $\mathbf{r}(u,v) = u\vec{i} + u\cos v\vec{j} + u\sin v\vec{k}$ with $0 \le u \le 4$ and $0 \le v \le \pi$.
a) Find the area S of σ. b) Find the mass M of σ if its density is $\delta(x,y,z) = x^2+y^2+z^2$.

a) $S = \iint_\sigma dS = \iint_R \|\vec{r}_u \times \vec{r}_v\| dA$, $R = [0,4]\times[0,\pi]$ in uv-plane

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & \cos v & \sin v \\ 0 & -u\sin v & u\cos v \end{vmatrix} = (u\cos^2 v + u\sin^2 v)\vec{i} - (u\cos v - 0)\vec{j} + (-u\sin v - 0)\vec{k} = u\vec{i} - u\cos v\vec{j} - u\sin v\vec{k}$$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{u^2 + u^2(\cos^2 v + \sin^2 v)} = \sqrt{2u^2} = \sqrt{2}u$; $S = \int_0^{\pi}\int_0^4 \sqrt{2}u \, du\, dv = \pi\sqrt{2}\left.\frac{u^2}{2}\right]_0^4$

$S = 8\pi\sqrt{2}$

b) $M = \iint_\sigma \delta \, dS = \int_0^{\pi}\int_0^4 (2u^2)\sqrt{2}u \, du\, dv$, since $x^2+y^2+z^2 = u^2 + u^2(\cos^2 v + \sin^2 v) = 2u^2$

$= \pi(2\sqrt{2})\left.\frac{u^4}{4}\right]_0^4 = 2\pi\sqrt{2}(4^3) = 128\pi\sqrt{2}$

5. [10] Evaluate the surface integral $\int\int_\sigma x\sqrt{z}dS$ where σ is the portion of the paraboloid $z = x^2 + y^2$ in the first octant between the planes $z = 0$ and $z = 4$.

$z_x = 2x$, $z_y = 2y$, $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4(x^2+y^2)+1} = \sqrt{4r^2+1}$, $x = r\cos\theta$, $y = r\sin\theta$

$0 \le \theta \le \frac{\pi}{2}$ (1st quadrant)

$\iint_\sigma x\sqrt{z}\,dS = \iint_R x\sqrt{x^2+y^2}\sqrt{4(x^2+y^2)+1}\,dA$, R = projection of σ on xy-plane = region in 1st quadrant enclosed by the circle $x^2+y^2 = 4$ and the lines $x=0$, $y=0$

polar $= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\sqrt{4r^2+1}\, r\,dr\,d\theta$

$= \int_0^{\pi/2}\cos\theta\,d\theta \cdot \int_0^2 r^3\sqrt{4r^2+1}\,dr$

$= [\sin\theta]_0^{\pi/2}\left\{\left. r^2\frac{(4r^2+1)^{3/2}}{12}\right]_0^2 - \frac{2}{12}\int_0^2 r(4r^2+1)^{3/2}dr\right\} = \frac{17\sqrt{17}}{3} - \frac{1}{120}(17^{5/2} - 1)$

$= 1\cdot\left(\frac{17}{3}^{3/2} - \frac{1}{6}(4r^2+1)^{5/2}/20\right]_0^2$

6. [12] Find all points on the plane $x+2y+3z=6$ that are closest to the origin, and state how far they are from the origin.

Let (x,y,z) be a point on the plane; $d((x,y,z),(0,0,0)) = \sqrt{x^2+y^2+z^2}$.
Set $f(x,y,z) = x^2+y^2+z^2$, d is a minimum if and only if f is a minimum
We shall find the minimum value of f subject to the constraint
$x+2y+3z=6$. We shall solve $\nabla f(x,y,z) = \lambda \nabla(x+2y+3z-6)$
So $\langle 2x, 2y, 2z\rangle = \lambda\langle 1,2,3\rangle \rightarrow x=\lambda/2,\ y=\lambda,\ z=3\lambda/2$
So $y=2x,\ z=3x$. $x+2(2x)+3(3x)=6 \rightarrow 14x=6 \rightarrow x=3/7,\ y=6/7,\ z=9/7$
The minimum value of f occurs at $(\frac{3}{7},\frac{6}{7},\frac{9}{7})$ with value $\frac{9+36+81}{49} = \frac{126}{49} = \frac{18}{7}$
So the minimum value $\sqrt{\frac{18}{7}}$ of the distance from a point on the plane to the origin occurs at $(\frac{3}{7},\frac{6}{7},\frac{9}{7})$.

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7. [15] Let $F(x,y) = (x^3y+4e^{-2x})\vec{i} + (\frac{x^4}{4}+y^2)\vec{j}$. a) Show that F is conservative. b) Find a potential function φ for F. c) Evaluate the line integral $\int_C (x^3y+4e^{-2x})\,dx + (\frac{x^4}{4}+y^2)\,dy$ along the curve C parametrized by $\vec{r}(t) = \cos^3 t\,\vec{i} + \sin^3 t\,\vec{j}$, $0 \le t \le \pi$.

a) $\partial_y(x^3y+4e^{-2x}) = x^3 = \partial_x(\frac{x^4}{4}+y^2)$; so F is conservative

b) (i) $\varphi_x = x^3y+4e^{-2x}$, (ii) $\varphi_y = \frac{x^4}{4}+y^2$.
Integrating (i) w.r.t. x: $\varphi(x,y) = \int(x^3y+4e^{-2x})dx = \frac{x^4}{4}y - 2e^{-2x} + k(y)$ (iii)
Differentiating (iii) w.r.t. y: $\varphi_y(x,y) = \frac{x^4}{4} + k'(y) = \frac{x^4}{4}+y^2$; so $k'(y)=y^2$,
and $k(y) = \frac{y^3}{3}$. Hence $\varphi(x,y) = \frac{x^4}{4}y - 2e^{-2x} + \frac{y^3}{3}$

$\int_C (x^3y+4e^{-2x})dx + (\frac{x^4}{4}+y^2)dy = \varphi(-1,0) - \varphi(1,0)$, since $\vec{r}(\pi) = (-1,0)$, $\vec{r}(0) = (1,0)$
by FTLI
$= 0 - 2e^2 + 0 - (0 - 2e^{-2} + 0)$
$= -2e^2 + 2e^{-2}$.