MAC 2313 (Calculus III) — Answers
Test 2, Wednesday February 25, 2015

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve any of the points assigned to any question. *You will not get any credit if you do not show the steps to your answers.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. 2 pages. Total=65 points. Good luck.**

1 . [20] a) Find an equation for the tangent plane and parametric equations for the normal line to the cone $z = 4 + \sqrt{x^2 + y^2}$ at the point (-1,$\sqrt{3}$,6).

Set $F(x,y,z) = z - 4 - \sqrt{x^2+y^2}$. Then $\nabla F(x,y,z) = \langle -\frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \rangle$

$\nabla F(-1,\sqrt{3},6) = \langle \frac{1}{2}, -\frac{\sqrt{3}}{2}, 1 \rangle$ = a normal to tangent plane at $(-1,\sqrt{3},6)$

Equation of tangent plane: $\frac{1}{2}(x+1) - \frac{\sqrt{3}}{2}(y-\sqrt{3}) + z - 6 = 0$

Parametric equations of normal line

$x = -1 + t, \quad y = \sqrt{3} - \sqrt{3}t, \quad z = 6 + 2t$, by using $2\nabla F(-1,\sqrt{3},6)$.

b) Let $f(x, y) = 2x^2 - 4xy + y^2 - 2x + 2$. Find all the critical points of f and classify each of them as a local maximum, a local minimum, or a saddle point.

$f_x(x,y) = 4x - 4y - 2 \qquad f_y(x,y) = -4x + 2y$

$f_x(x,y) = 0 \to \underbrace{2x - 2y = 1}_{(1)}, \quad f_y(x,y) = 0 \to \underbrace{2x = y}_{(2)}$

using (2) in (1) yields: $2x - 4x = 1$ or $-2x = 1$; so $x = -1/2$ and $y = -1$. The only critical point is $(-1/2, -1)$. $f_{xx}(x,y) = 4$,

$f_{xy}(x,y) = -4$, $f_{yy}(x,y) = 2$

$\Delta(-1/2,-1) = f_{xy}(-1/2,-1)^2 - f_{xx}(-\frac{1}{2},-1) f_{yy}(-\frac{1}{2},-1) = 16 - 4(2) = 16 - 8 = 8 > 0$

So f has a saddle point at $(-\frac{1}{2}, -1)$.

2. [10] Let $f(x,y) = \begin{cases} \dfrac{y^{\frac{12}{5}}x}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

a) Find the partial derivatives $f_x(0,0)$, and $f_y(0,0)$.

$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$

b) Is f differentiable at $(0,0)$?

$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - h f_x(0,0) - k f_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{k^{\frac{12}{5}}h}{(h^2+k^2)^{3/2}}$; set $h = r\cos\theta$, $k = r\sin\theta$, $r > 0$, $0 \le \theta < 2\pi$.

$= \lim_{r\to 0^+} \frac{r^{\frac{12}{5}}(\cos\theta)^{\frac{12}{5}} r\sin\theta}{r^3} = \lim_{r\to 0^+} r^{\frac{17}{5}-3}(\cos\theta)^{\frac{12}{5}}\sin\theta$

$= \lim_{r\to 0^+} r^{\frac{2}{5}}(\cos\theta)^{\frac{12}{5}}\sin\theta = 0$, for all θ; so f is differentiable at $(0,0)$.

3. [8] Let $f = f(u, v)$ be a differentiable function. Set $u = x^2 - y^2$, $v = xy$, and $g(x, y) = f(u, v)$. Use the chain rule to find $g_x(x, y)$ and $g_y(x, y)$.

$g_x(x,y) = f_u(u,v)u_x + f_v(u,v)v_x = 2x f_u(u,v) + y f_v(u,v)$, as $u_x = 2x$, $v_x = y$

$g_y(x,y) = f_u(u,v)u_y + f_v(u,v)v_y = -2y f_u(u,v) + x f_v(u,v)$, as $u_y = -2y$, $v_y = x$

4. [9] Let $f(x, y, z) = zye^{z+yx}$. a) Find the gradient of f at $P(-1, 1, 1)$. b) Find a unit vector in the direction in which f decreases most rapidly at the point $P(-1, 1, 1)$, and find the rate of change of f at P in that direction. c) Find the directional derivative of f in the direction of the vector $\vec{v}=-\vec{i}+2\vec{j}-3\vec{k}$ at the point $A(-1, -1, 1)$

a) $\nabla f(x,y,z) = \langle zy^2 e^{z+yx}, (z+zyx)e^{z+xy}, (y+zy)e^{z+xy} \rangle$

$\nabla f(P) = \langle 1, 0, 2 \rangle$

b) $\dfrac{-\nabla f(P)}{\|\nabla f(P)\|} = \dfrac{\langle -1, 0, -2 \rangle}{\sqrt{5}}$; rate of change $= -\|\nabla f(P)\| = -\sqrt{5}$

c) $D_{\vec{v}} f(A) = \nabla f(A) \cdot \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{e^2 \langle 1, 2, -2 \rangle \cdot \langle -1, 2, -3 \rangle}{\sqrt{14}} = \dfrac{(-1+4+6)e^2}{\sqrt{14}}$

$= \dfrac{9e^2}{\sqrt{14}}$

5. [8] Let $h(x, y, z) = xy - 2yz + zx$. Find the local linear approximation for h about the point $P(1, 1, 1)$, and use it to approximate $h(0.99, 1.01, 0.98)$.

$h_x(x,y,z) = y + z$, $h_y(x,y,z) = x - 2z$, $h_z(x,y,z) = -2y + x$

$h(P) = 1 - 2 + 1 = 0$, $h_x(P) = 2$, $h_y(P) = -1$, $h_z(P) = -1$

$L(x,y,z) = h(P) + h_x(P)(x-1) + h_y(P)(y-1) + h_z(P)(z-1)$

$= 2(x-1) - (y-1) - (z-1)$

$h(0.99, 1.01, 0.98) \approx 2(0.99-1) - (1.01-1) - (0.98-1)$

$\approx 2(-0.01) - 0.01 - (-0.02)$

≈ -0.01

6. [10] Mark each statement true or false. No explanation is needed.

a) If f is continuous at $(1, -2)$, then f is differentiable at $(1, -2)$. F, pick $f(x,y) = \sqrt{(x-1)^2 + (y+2)^2}$; f is cont but not diff-ble

b) If $f(x, y) \to 2$ as (x, y) approaches $(2, 2)$ along the line $y = 2$ and $f(x, y) \to 2$ as (x, y) approaches $(2, 2)$ along the smooth curve $y = 2 - (x - 2)^3$, then $\lim_{(x,y)\to(2,2)} f(x, y) = 2$. False

c) If $z(t) = f(x(t), y(t))$, then $\frac{dz}{dt} = f_x(\frac{dx}{dt}, y) + f_y(x, \frac{dy}{dt})$. False (See Theorem 13.5.1 in text)

d) If $f_x(1, 1)$ and $f_y(1, 1)$ both exist, then f is continuous at (1,1). False, pick $f(x,y) = \begin{cases} \dfrac{(x-1)(y-1)}{(x-1)^2+(y-1)^2} & \text{if } (x,y) \neq (1,1) \\ 0, & \text{if } (x,y) = (1,1) \end{cases}$

e) If $f = f(x, y, z)$ is differentiable at the point $B(3, 1, 2)$, then the directional derivative of f at B in the direction of the vector $\vec{u}=-2\vec{i}+\vec{j}-\vec{k}$ is given by $D_{\vec{u}} f(B) = \nabla f(B) \cdot \vec{u}$. False; $D_{\vec{u}} f(B) = \nabla f(B) \cdot \dfrac{\vec{u}}{\|\vec{u}\|}$ according to our definition of $D_{\vec{u}} f(B)$.