MAP 2302 (Differential Equations) — Key
TEST 2, Thursday March 25, 2010

Name:

PID:

Remember that no documents or calculators are allowed during the test. You shall show all your work to deserve the full mark assigned to any question. Good luck.

1. [20] Given that $y = \sin(2x)$ solves the differential equation: $y^{\text{iv}} - 2y''' + 6y'' - 8y' + 8y = 0$, find the general solution.

Since $\sin(2x)$ solves the DE., $2i$ and $-2i$ are roots of the auxiliary equation: $m^4 - 2m^3 + 6m^2 - 8m + 8 = 0$;
So $m^4 - 2m^3 + 6m^2 - 8m + 8 = (m-2i)(m+2i)$(Quadratic polynomial)
$= (m^2+4)(m^2 - 2m + 2)$

Now the roots of $m^2 - 2m + 2 = 0$ are

$$m_3 = \frac{2+\sqrt{2^2-4(2)}}{2} = 1 + \frac{\sqrt{-4}}{2} = 1+i, \quad \text{and } m_4 = \frac{2-\sqrt{2^2-8}}{2} = 1-i$$

general solution: $y = C_1 \sin(2x) + C_2 \cos(2x) + e^{x}(C_3 \sin x + C_4 \cos x)$
C_1, C_2, C_3, C_4 = arbitrary constants.

2. [20] Use the method of undetermined coefficients to solve the differential equation:
$y'' + 6y' + 13y = e^x \sin x$

Homogeneous equation: $y'' + 6y' + 13y = 0$

Auxiliary equation: $m^2 + 6m + 13 = 0$

$$m^2 + 6m + 9 + 4 = 0$$
$$(m+3)^2 - (2i)^2 = 0$$
$$(m+3-2i)(m+3+2i) = 0$$
$$m_1 = -3+2i, \quad m_2 = -3-2i$$

$y_c = e^{-3x}(c_1 \sin(2x) + c_2 \cos(2x))$, c_1, c_2 = arbitrary constants

Seek a particular solution of given D.E. in the form

$y_p = (a \sin x + b \cos x)e^x$.

So
$$y_p' = (a\cos x + (-b)\sin x + a \sin x + b\cos x)e^x$$
$$= ((a+b)\cos x + (a-b)\sin x)e^x$$
$$y_p'' = (-(a+b)\sin x + (a-b)\cos x + (a+b)\cos x + (a-b)\sin x)e^x$$
$$= (-2b\sin x + 2a\cos x)e^x$$

$$y_p'' + 6y_p' + 13y_p = (-2b\sin x + 2a\cos x + 6(a+b)\cos x + 6(a-b)\sin x + 13a\sin x + 13b\cos x)e^x$$
$$= ((19a - 8b)\sin x + (8a + 19b)\cos x)e^x$$
$$= e^x \sin x$$

Hence
$$19a - 8b = 1 \quad (1)$$
$$8a + 19b = 0 \quad (2)$$

$19\cdot(1) + 8\cdot(2)$ yields: $(19^2 + 8^2)a = 19 \rightarrow a = \frac{19}{19^2+8^2}$

$-8\cdot(1) + 19\cdot(2)$ yields: $(8^2 + 19^2)b = -8 \rightarrow b = \frac{-8}{19^2+8^2}$

General solution: $y = e^{-3x}(c_1 \sin(2x) + c_2\cos(2x)) + e^x\left(\frac{19}{19^2+8^2}\sin x - \frac{8}{19^2+8^2}\cos x\right)$.

3. [10+20] a) Solve the Cauchy-Euler equation: $x^2y'' - xy' + 4y = 0$. Write the general solution in terms of the variable $x > 0$. b) Use the method of variation of parameters to find the general solution of the differential equation: $y'' + 4y' + 4y = x^{-4}e^{-2x}$.

a) $x = e^t$, $\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} = x\frac{dy}{dx}$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{dx}{dt}\frac{dy}{dx} + x\cdot\frac{dx}{dt}\frac{d^2y}{dx^2}$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

Therefore D.E. becomes

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 4y = 0.$$ Aux. eqn: $m^2 - 2m + 4 = 0$

$$\Delta = (-2)^2 - 4(4) = -12$$

$$m_1 = \frac{2 + i\sqrt{12}}{2} = 1 + i\sqrt{3}$$

$$m_2 = \frac{2 - i\sqrt{12}}{2} = 1 - i\sqrt{3}$$

$$y_c = (C_1\sin(\sqrt{3}t) + C_2\cos(\sqrt{3}t))e^t$$

$$= (C_1\sin(\sqrt{3}\ln x) + C_2\cos(\sqrt{3}\ln x))x,$$ C_1, C_2 = arbitrary constants

b) Homogeneous eqn: $y'' + 4y' + 4y = 0$. Aux. eqn: $m^2 + 4m + 4 = 0$

$$(m+2)^2 = 0$$

$$m_1 = m_2 = -2$$ 2

$$y_c = (C_1 + C_2x)e^{-2x},$$ 3 C_1, C_2 = arbitrary constants.

Seek a particular solution in the form: $y_p = v_1(x)e^{-2x} + v_2(x)xe^{-2x}$ 2 with

$$v_1'e^{-2x} + v_2'xe^{-2x} = 0$$ 2

$$-2v_1'e^{-2x} + (1-2x)v_2'e^{-2x} = x^{-4}e^{-2x}$$ 3

or

$$\begin{cases} v_1' + xv_2' = 0 & (1) \\ -2v_1' + (1-2x)v_2' = x^{-4} & (2) \end{cases}$$

$2\cdot(1) + (2)$ yields: $v_2' = x^{-4}$ 1.5 $\rightarrow v_2 = \int x^{-4}dx = -\frac{x^{-3}}{3}$ 1.5

$(1) \rightarrow v_1' = -xv_2' = -x(x^{-4}) = -x^{-3}$ 1.5 $\rightarrow v_1 = \int -x^{-3}dx = \frac{x^{-2}}{2}$ 1.5

$$y_p = \frac{x^{-2}}{2}e^{-2x} - x\left(\frac{x^{-3}}{3}\right)e^{-2x} = x^{-2}\left(\frac{1}{2} - \frac{1}{3}\right)e^{-2x} = \frac{x^{-2}}{6}e^{-2x}$$

General solution: $y = \left(C_1 + C_2x + \frac{x^{-2}}{6}\right)e^{-2x}$.

4. [15] Find power series solutions in powers of x of the differential equation: $y'' + (x+1)y' + y = 0$. Write down the general solutions including powers of x up to x^3.

Seek $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) x^{n-2} c_n$

Consequently:

$y'' + (x+1)y' + y = 0$ becomes

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

Replace $n-2$ by n — Replace $n-1$ by n

8

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) c_n x^n + \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n = 0$$

or

$$\sum_{n=0}^{\infty} (n+1)\left[(n+2) c_{n+2} + c_{n+1} + c_n\right] x^n = 0,$$ for all x close enough to zero.

Hence

2 $(n+1)\left[(n+2)c_{n+2} + c_{n+1} + c_n\right] = 0$ or $c_{n+2} = -\frac{(c_{n+1} + c_n)}{n+2}$, $n = 0, 1, 2, \ldots$

$n=0$: $c_2 = -\frac{c_1}{2} - \frac{c_0}{2}$ 1

$n=1$: $c_3 = -\frac{c_2}{3} - \frac{c_1}{3} = \frac{c_0}{6} + \frac{c_1}{6} - \frac{c_1}{3} = \frac{c_0}{6} - \frac{c_1}{6}$ 1

1.5

$$y = c_0\left(1 - \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) + c_1\left(x - \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)$$

c_0, c_1 = arbitrary constants. 1.5

5. [15] Use the method of Frobenius to find two linearly independent series solutions of the form $x^r \sum_{n=0}^{\infty} a_n x^n$ to the differential equation equation $2x^2y'' + 3xy' + (4x-6)y = 0, \quad 0 < x < R.$
(First find and solve the indicial equation, then for each indicial root, find a recurrence relation between a_n, and a_{n-1}. This will be enough.)

Seek $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Then $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

D.E. becomes:

$$2\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + 3\sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \underbrace{4\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{\text{Replace } n+1 \text{ by } n} - 6\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

8

or

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) + 3(n+r) - 6] a_n x^{n+r} + 4\sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

or

$$(2r(r-1) + 3r - 6) a_0 x^r + \sum_{n=1}^{\infty} \{[2(n+r)(n+r-1) + 3(n+r) - 6] a_n + 4a_{n-1}\} x^{n+r} = 0$$

for all x in $0 < x < R$

Hence

Indicial eqn: $(2r(r-1) + 3r - 6) a_0 = 0$, $a_0 \neq 0$

and

(*) 2 $(2(n+r)(n+r-1) + 3(n+r) - 6) a_n + 4a_{n-1} = 0$, $n = 1, 2, \ldots$

Indicial eqn reduces to $2r^2 + r - 6 = 0$ or $(2r-3)(r+2) = 0$

$r_1 = 3/2$, $r_2 = -2$ 2

For $r = 3/2$, (*) becomes:

$$[(2n+3)(n+\tfrac{1}{2}) + 3(n+\tfrac{3}{2}) - 6] a_n + 4a_{n-1} = 0, \quad n = 1, 2, \ldots$$

For $r = -2$, (*) becomes:

$$[2(n-2)(n-3) + 3(n-2) - 6] a_n + 4a_{n-1} = 0, \quad n = 1, 2, \ldots$$