MAP 2302 (Differential Equations) — Key
TEST 2, Friday April 11, 2014

Name:

PID:

1. [20] Find power series solutions in powers of x of the differential equation: $y'' - x^2y' + 3y = 0$. Write down the general solution including powers of x up to x^4.

We seek $y = \sum_{n=0}^{\infty} c_n x^n$. So $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$.

Reporting that in D.E., we find!

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} n c_n x^{n+1} + 3\sum_{n=0}^{\infty} c_n x^n = 0$$

Replace $n-2$ by n

We can start at $n=0$ since that adds no additional term, and all powers of x are positive

Replace $n+1$ by n

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} (n-1)c_{n-1} x^n + 3\sum_{n=0}^{\infty} c_n x^n = 0$$

Take out the first term in 1st and 3rd sums and gather remaining sums;

$$2c_2 + 3c_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} + 3c_n - (n-1)c_{n-1}]x^n = 0$$ for all x in some small interval $|x|<R$.

So

$$2c_2 + 3c_0 = 0 \rightarrow c_2 = -\frac{3}{2}c_0$$

$$(n+2)(n+1)c_{n+2} + 3c_n - (n-1)c_{n-1} = 0, \quad n = 1, 2, 3, \ldots$$

$n=1$: $6c_3 + 3c_1 = 0 \rightarrow c_3 = -\frac{c_1}{2}$

$n=2$: $12c_4 + 3c_2 - c_1 = 0 \rightarrow c_4 = -\frac{c_2}{4} + \frac{c_1}{12} = \frac{3}{8}c_0 + \frac{c_1}{12}$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$
$$= c_0\left(1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \ldots\right) + c_1\left(x - \frac{x^3}{2} + \frac{x^4}{12} + \ldots\right)$$

2. [20] Use the method of Frobenius to solve the differential equation: $2x^2y'' - 2xy' + (x+2)y = 0, \quad 0 < x < R$. a) Find the indicial equation, and solve it. b) Write down the recursion formula relating the coefficients. c) Find the general solution y_1 corresponding to the root of the indicial equation including powers of x up to x^3. d) Write down the form of another solution y_2 that is linearly independent of y_1.

We seek $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$. So $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$. Reporting that in D.E.:

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Gathering the first two sums and the last one, we get

$$2\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) - (n+r) + 1\right]a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Replace $n+1$ with n

$$2\sum_{n=0}^{\infty}\left[(n+r)(n+r-2)+1\right]a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0 \text{ for all } 0<x<R, \ R \text{ small}$$

Taking out the first term in the left sum, and gathering the remaining sums, we find:

$$2(r(r-2)+1)a_0 x^r + \sum_{n=1}^{\infty}\left\{2[(n+r)(n+r-2)+1]a_n + a_{n-1}\right\} x^{n+r} = 0$$

Hence all coefficients are zero. So

$2(r(r-2)+1)a_0 = 0$ (Indicial equation) $\rightarrow r^2 - 2r + 1 = 0$ or $(r-1)^2 = 0$

So $r_1 = r_2 = 1$

b) $2[(n+r)(n+r-2)+1]a_n + a_{n-1} = 0, \quad n = 1, 2, \ldots$

or $2[(n+1)(n-1)+1]a_n + a_{n-1} = 0, \quad n = 1, 2, 3, \ldots$

c) $n=1$: $2a_1 + a_0 = 0 \rightarrow a_1 = -a_0/2$

$n=2$: $8a_2 + a_1 = 0 \rightarrow a_2 = -a_1/8 = a_0/16$

$n=3$: $18a_3 + a_2 = 0 \rightarrow a_3 = -a_2/18 = -a_0/16(18)$

$$y_1 = c_0 x\left(1 - \frac{x}{2} + \frac{x^2}{16} - \frac{x^3}{16(18)} + \cdots\right)$$

d) $$y_2 = x^2 \sum_{n=0}^{\infty} c_n^* x^n + y_1(x)\ln x, \quad c_0^* \neq 0$$

Note: The general solution of the D.E is given by

$y = Ay_1 + By_2$, where $c_0 = 1$ and $c_0^* = 1$, and A, B are arbitrary constants

3. [25] Solve the Cauchy-Euler differential equation: $x^2y'' + 3xy' + 2y = 10\cos(\ln x)$, $x > 0$. (Show all your work)

Set $z(t) = y(x)$, $x = e^t$, $t = \ln x$, $\frac{dx}{dt} = e^t = x$

$\frac{dz}{dt} = \frac{dx}{dt}\frac{dy}{dx} = x\frac{dy}{dx}$; $\frac{d^2z}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{dx}{dt}\frac{dy}{dx} + x\frac{dx}{dt}\frac{d^2y}{dx^2}$

$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$

$$\frac{d^2z}{dt^2} = x^2y'' + xy'$$
$$= x^2y'' + 3xy' + 2y - 2xy' - 2y$$
$$= 10\cos(t) - 2\frac{dz}{dt} - 2z$$

$$\frac{d^2z}{dt^2} + 2\frac{dz}{dt} + 2z = 10\cos(t)$$

Auxiliary equ: $m^2 + 2m + 2 = 0$ or $m^2 + 2m + 1 + 1 = 0$ or $(m+1)^2 + 1 = 0$

So $(m+1)^2 = -1 \rightarrow m+1 = \pm i$, $m_1 = -1 - i$, $m_2 = -1 + i$

$z_c = (c_1\cos(t) + c_2\sin(t))e^{-t}$, $c_1, c_2 =$ constants.

Use method of undetermined coefficients to get a particular soln z_p

$S_{\cos(t)} = \{\cos(t), \sin(t)\}$. Seek $z_p = A\cos t + B\sin t$. So

$z_p' = -A\sin t + B\cos t$, $z_p'' = -A\cos t - B\sin t$. Reporting that in D.E;

$$-A\cos t - B\sin t + 2(-A\sin t + B\cos t) + 2A\cos t + 2B\sin t = 10\cos t$$

$$A\cos t + 2B\cos t + (B - 2A)\sin t = 10\cos t$$

So $A + 2B = 10$

$B - 2A = 0 \rightarrow B = 2A$; So $A + 2(2A) = 10 \rightarrow 5A = 10 \rightarrow A = 2$

$B = 2(2) = 4$

$z_p = 2\cos t + 4\sin t$

The general soln of D.E (Cauchy-Euler equ) is

$$y = (c_1\cos(\ln x) + c_2\sin(\ln x))x^{-1} + 2\cos(\ln x) + 4\sin(\ln x).$$

4. [20] (Remember that $\mathcal{L}(t^n)(s) = n!/s^{n+1}$, $\mathcal{L}(\sin(bt))(s) = b/(s^2+b^2)$, $\mathcal{L}(\cos(bt))(s) = s/(s^2+b^2)$), $\mathcal{L}(e^{at})(s) = 1/s - a$.)
Use the Laplace transform to solve the initial-value problem: (Show all your work),
$y'' - y' - 12y = 11$
$y(0) = 0, \quad y'(0) = 0.$

Set $Y = \mathcal{L}y$

$$\mathcal{L}(y'' - y' - 12y)(s) = \mathcal{L}(11)(s) = \frac{11}{s}$$

$$s^2Y - sy(0) - y'(0) - (sY - y(0)) - 12Y = \frac{11}{s}$$

$$(s^2 - s - 12)Y = \frac{11}{s}$$

$$(s-4)(s+3)Y = \frac{11}{s}$$

$$Y = \frac{11}{s(s-4)(s+3)} = \frac{a}{s} + \frac{b}{s-4} + \frac{c}{s+3}$$

To get a, multiply both sides by s, simplify and set $s=0$: $\frac{11}{-12} = a$
For b, do the same with multiplication by $s-4$, then set $s=4$: $\frac{11}{28} = b$
For c, — " — " — " — ", $s+3$, — " — " $s=-3$: $\frac{11}{21} = c$

$$Y = \frac{-11/12}{s} + \frac{11/28}{s-4} + \frac{11/21}{s+3}$$

$$y(t) = \mathcal{L}^{-1}Y(t) = -\frac{11}{12} + \frac{11}{28}e^{4t} + \frac{11}{21}e^{-3t}$$

5. [15] Find the Laplace transform of the function $f(t) = \begin{cases} -2, & 0 < t < 3 \\ 7, & 3 < t < 8 \\ -3, & t > 8 \end{cases}$

$$\begin{aligned} f(t) &= -2(u_0(t) - u_3(t)) + 7(u_3(t) - u_8(t)) - 3u_8(t) \\ &= -2 + (2u_3 + 7u_3 - 7u_8 - 3u_8)(t) \\ &= -2 + 9u_3(t) - 10u_8(t) \end{aligned}$$

Hence $\mathcal{L}f(s) = -2\mathcal{L}(1)(s) + 9\mathcal{L}(u_3)(s) - 10\mathcal{L}(u_8)(s)$

$$= -\frac{2}{s} + 9\frac{e^{-3s}}{s} - 10\frac{e^{-8s}}{s}$$