MAP 2302 (Differential Equations) — Answers
TEST 2, Friday March 9, 2018

Name: PID:

Remember that no documents or calculators are allowed during the test. You must show all your work to deserve the full credit assigned to any question. 3 pages.

1.[10] a) Show that the two functions e^x and xe^{-x} are linearly independent on the interval $(-\infty, 0]$.

$$W(e^x, xe^{-x}) = \begin{vmatrix} e^x & xe^{-x} \\ e^x & (1-x)e^{-x} \end{vmatrix}$$
$$= (1-x)e^x \cdot e^{-x} - x e^x \cdot e^{-x}$$
$$= 1 - x - x, \text{ as } e^x e^{-x} = e^{x-x} = e^0 = 1$$
$$= 1 - 2x \neq 0 \text{ for all } x \text{ in } (-\infty, 0]$$

So e^x and xe^{-x} are linearly independent on $(-\infty, 0]$.

b) Given that 0,0,0,3,1-2i,1+2i,1-2i,1+2i,1+$\sqrt{2}$,1-$\sqrt{2}$, 9i, -9i are the roots of the auxiliary equation corresponding to some 12th-order homogeneous linear differential equation with constant coefficients, write down the general solution of the differential equation.

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{3x} + e^x\left((c_5 + c_6 x)\cos(2x) + (c_7 + c_8 x)\sin(2x)\right)$$
$$+ c_9 e^{(1+\sqrt{2})x} + c_{10} e^{(1-\sqrt{2})x} + c_{11}\cos(9x) + c_{12}\sin(9x)$$

$c_1, c_2, \ldots, c_{12}$ = constants.

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2. [10] Transform the Cauchy-Euler equation: $-3x^2y'' + 7xy' - 5y = 12x^3, \quad x > 0$, into a differential equation in the variable t by setting $x = e^t$. You must show all the steps, but do not solve the differential equation in the variable t.

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}, \text{ as } t = \ln x, \text{ and } \frac{dt}{dx} = \frac{1}{x}; \text{ So } x\frac{dy}{dx} = \frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{1}{x}\right)\cdot\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

So $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$

The Cauchy-Euler equation becomes

$$-3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 7\frac{dy}{dt} - 5y = 12(e^t)^3 = 12e^{3t}$$

$$-3\frac{d^2y}{dt^2} + 10\frac{dy}{dt} - 5y = 12e^{3t}$$

3. [20] Use the method of undetermined coefficients to solve the differential equation: $y'' + 9y = 3x^2 - 4x$.

Homogeneous D.E:

$y'' + 9y = 0$

Aux. equ: $m^2 + 9 = 0 \rightarrow m_1 = -3i, \; m_2 = 3i$

$y_c = C_1 \cos(3x) + C_2 \sin(3x)$, $C_1, C_2 =$ constant

UC sets

$S_{x^2} = \{x^2, x, 1\}$, $S_x = \{x, 1\}$

S_x is contained in S_{x^2}; so discard S_x.

$S = S_{x^2} = \{x^2, x, 1\}$. Seek $y_p = Ax^2 + Bx + C$, $A, B, C =$ constants to be determined

$y_p' = 2Ax + B$, $y_p'' = 2A$

$$\begin{aligned} y_p'' + 9y_p &= 2A + 9Ax^2 + 9Bx + 9C \\ &= 2A + 9C + 9Bx + 9Ax^2 \\ &= 3x^2 - 4x \end{aligned}$$

So

$2A + 9C = 0 \rightarrow C = -2A/9$

$9B = -4 \rightarrow B = -4/9$ $\rightarrow C = -\frac{2}{27}$

$9A = 3 \rightarrow A = 1/3$

$$y_p = \frac{x^2}{3} - \frac{4}{9}x - \frac{2}{27}$$

$$y = C_1 \cos(3x) + C_2 \sin(3x) + \frac{x^2}{3} - \frac{4}{9}x - \frac{2}{27}$$

4. [20] Use the variation of parameters method to solve the differential equation:
$y'' - 2y' + y = \frac{e^x}{1+x^2}$

Homogeneous D.E

$y'' - 2y' + y = 0$

Aux. equation:

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0, \quad m_1 = 1 = m_2$

$y_c = (c_1 + c_2 x) e^x$

Seek $y_p = v_1(x) e^x + v_2(x) x e^x$ with

$v_1' e^x + v_2' x e^x = 0 \longrightarrow v_1' + x v_2' = 0 \quad (1)$

$v_1' e^x + v_2'(x+1) e^x = \frac{e^x}{1+x^2} \quad v_1' + (x+1) v_2' = \frac{1}{1+x^2} \quad (2)$

Subtract (1) from (2) to get

$v_2' = \frac{1}{1+x^2} \longrightarrow v_2(x) = \int \frac{dx}{1+x^2} = \arctan(x) = \tan^{-1} x$

$(1) \rightarrow v_1' = -x v_2' = -\frac{x}{1+x^2}$

$v_1 = \int -\frac{x}{1+x^2} dx = -\frac{1}{2} \ln(1+x^2)$, set $u = 1+x^2$
$du = 2x\,dx$ or
$x\,dx = \frac{1}{2} du$

Hence $y_p = -\frac{e^x}{2} \ln(1+x^2) + x e^x \tan^{-1} x$

General soln of DE $y = y_c + y_p$

$= (c_1 + c_2 x) e^x - \frac{e^x}{2} \ln(1+x^2) + x e^x \tan^{-1} x.$