MAS 3105 (Linear Algebra)
Test 2, Friday June 06, 2014

Name: **PID:**

Remember that you won't get any credit if you do not show the steps to your answers.

1. [20] Find a basis for the null space and a basis for the column space of the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 5 & -3 \\ 2 & 4 & 6 & 8 \end{pmatrix}$.

For the null space, solve $Ax = 0_{\mathbb{R}^3}$ for $x \in \mathbb{R}^4$. We use the Gauss elimination process.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 5 & -3 \\ 2 & 4 & 6 & 8 \end{pmatrix} \xrightarrow[r_1+r_2]{-2r_1+r_3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2/4} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$Ax = 0_{\mathbb{R}^3}$ is equivalent to $x_1 + 2x_2 = -3x_3 - 4x_4$

$x_2 = -2x_3 - \frac{x_4}{4}$; $x_1 = -2(-2x_3 - \frac{x_4}{4}) - 3x_3 - 4x_4 = x_3 - \frac{7}{2}x_4$

$N(A) = \{x_3(1,-2,1,0)^T + x_4(-\frac{7}{2}, -\frac{1}{4}, 0, 1)^T; x_3 \in \mathbb{R}, x_4 \in \mathbb{R}\}$

Hence $\{(1,-2,1,0)^T, (-\frac{7}{2}, -\frac{1}{4}, 0, 1)^T\}$ is a basis for $N(A)$

Since x_1 and x_2 are the lead variables,
$\{(1,-1,2)^T, (2,2,4)^T\}$ is a basis for the column space of A.

<u>Note</u>: The REF or RREF of A and A have the same row space, but they do not, in general, have the same column space.

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2. [20] State whether each of the following statement is true or false. No explanations needed.

a) If U and V are nonempty subsets of a vector space W, then $U+V$ is a subspace of W. False (See Definition of subspace)

b) If A is a 12×11 matrix, then A and A^T have the same null space. False $N(A) \subseteq \mathbb{R}^{11}$ while $N(A^T) \subseteq \mathbb{R}^{12}$

c) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_9$ are vectors in a vector space E, and $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_9) = \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_8)$, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_9$ are linearly dependent. True; x_9 is a linear combination of $x_1, x_2, \ldots, x_8$.

d) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ span $\mathbb{R}^m$, then they are linearly independent. True, by Theorem 3.4.3

e) Let A, and C be arbitrary $n \times n$ matrices. If A is similar to C, then A^T is similar to C^T. True, Section 4.3, Pb 12(a)

f) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly independent, then they span $\mathbb{R}^m$. True, by Theorem 3.4.3

g) If L is a linear operator on $\mathbb{R}^n$ and $\mathbf{x}$ and $\mathbf{z}$ are two vectors in $\mathbb{R}^n$ with $L(\mathbf{x}) = L(\mathbf{z})$, then $\mathbf{x} = \mathbf{z}$. False

h) If U is the reduced row echelon form of a 10×14 matrix A, then U and A have the same row space. True, (Theorem 3.6.1)

i) If L is a linear operator on $\mathbb{R}^6$, then the range of L satisfies: $R(L) = \mathbb{R}^6$. False; Define $L(x) = 0_{\mathbb{R}^6}$, then $R(L) = \{0_{\mathbb{R}^6}\}$

j) If A, B, and C are $n \times n$ matrices such that A is similar to B and B is similar to C, then A is similar to C. True (4.3, Pb 7)

g) Pick $L: \mathbb{R}^3 \to \mathbb{R}^3$ by $L((x_1,x_2,x_3)^T) = (x_2, x_1, 0)^T$,
$x = (1,2,-1)^T$, $z = (1,2,3)^T$. $L(x) = (2,1,0)^T = L(z)$, but $x \neq z$

i) Pick $L: \mathbb{R}^6 \to \mathbb{R}^6$ by $L((x_1,x_2,x_3,x_4,x_5,x_6)^T) = ((x_1,x_2,x_3,0,0,0)^T)$
Then $R(L) = \mathbb{R}^3 \neq \mathbb{R}^6$

3. [10] Let $P_2 = \{a + bx + cx^2; a,\ b,\ c, x \in \mathbb{R}\}$ be the space of polynomials with degree less than three. Define on P_2 a mapping L by $L(p)(x) = 2p(x) - 3p'(x)$, where p' stands for the derivative function of p. Show that L is linear, and find the matrix representation A of L with respect to the ordered basis $\{1 - x, x^2, 1\}$.

Let $p, q \in P_2$, $\alpha \in \mathbb{R}$. $L(p+\alpha q)(x) = 2(p+\alpha q)(x) - 3(p+\alpha q)'(x)$

$= 2p(x) + 2\alpha q(x) - 3p'(x) - 3\alpha q'(x)$

$= 2p(x) - 3p'(x) + \alpha(2q(x) - 3q'(x))$

$= L(p)(x) + \alpha L(q)(x)$; so L is linear.

$L(1-x) = 2(1-x) - 3(-1)$

$= 2(1-x) + 0 \cdot x^2 + 3(1)$

$L(x^2) = 2x^2 - 3(2x) = 2x^2 - 6x$

$= 6(1-x) + 2x^2 - 6(1)$

$L(1) = 2(1) = 0 \cdot (1-x) + 0 \cdot x^2 + 2(1)$

$$A = \begin{pmatrix} 2 & 6 & 0 \\ 0 & 2 & 0 \\ 3 & -6 & 2 \end{pmatrix}$$

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4. [10] Find a basis for the subspace S of $\mathbb{R}^4$ given by $S = \{(a+3b+7c, 2a+b+4c, -2b-4c, -2c-2a)^T; a, b, c \in \mathbb{R}\}$.

If a vector $u \in S$, $u = a\underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix}}_{S_1} + b\underbrace{\begin{pmatrix} 3 \\ 1 \\ -2 \\ 0 \end{pmatrix}}_{S_2} + c\underbrace{\begin{pmatrix} 7 \\ 4 \\ -4 \\ -2 \end{pmatrix}}_{S_3}$

$S = \text{Span}(S_1, S_2, S_3)$.

Are S_1, S_2, S_3 lin. indep.t? Notice $S_3 = S_1 + 2S_2$. so $S = \text{Span}(S_1, S_2)$ and S_1 and S_2 are linearly independent; hence S_1, S_2 form a basis for S. You may also use Gauss elimination process to find that x_1, x_2 are the lead variables.

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5. [25] Let $\mathbf{u}_1 = (1, -1, 2)^T$, $\mathbf{u}_2 = (-1, 2, 3)^T$ and $\mathbf{u}_3 = (4, 1, -1)^T$ be vectors in $\mathbb{R}^3$. Let L be the linear operator defined on $\mathbb{R}^3$ by $L(x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3) = (2x_2 + x_1 - x_3)\mathbf{u}_1 + (2x_3 + x_2 - x_1)\mathbf{u}_2 + (2x_1 - x_2 + x_3)\mathbf{u}_3$. a)Show that $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ form a basis for $\mathbb{R}^3$. b) Find the transition matrix S from the ordered basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ to the standard basis $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$. c) Find the matrix B of L with respect to the ordered basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. d) Write down the form of the matrix A of L with respect to the standard basis $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$. (Dot not attempt to evaluate A.)

a) Thanks to Theorem 3.4.3, it suffices to show that u_1, u_2 and u_3 are linearly independent.

$$\det(u_1, u_2, u_3) = \begin{vmatrix} 1 & -1 & 4 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{vmatrix} \xrightarrow[-2r_1+r_3]{r_1+r_2} \begin{vmatrix} 1 & -1 & 4 \\ 0 & 1 & 5 \\ 0 & 5 & -9 \end{vmatrix} \xrightarrow{-5r_2+r_3} \begin{vmatrix} 1 & -1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -34 \end{vmatrix} = -34 \neq 0$$

So u_1, u_2 and u_3 are linearly independent, and they span $\mathbb{R}^3$ by Th. 3.4.3; Hence u_1, u_2 and u_3 form a basis for $\mathbb{R}^3$.

b) $$S = \begin{pmatrix} 1 & -1 & 4 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{pmatrix}$$

c) $L(u_1) = u_1 - u_2 + 2u_3$, since $u_1 = u_1 + 0u_2 + 0u_3$; Similarly,

$L(u_2) = 2u_1 + u_2 - u_3$

$L(u_3) = -u_1 + 2u_2 + u_3$

$$B = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

d) $A = SBS^{-1}$, where S is the transition matrix in b).

6. [15] a) Let A and B be 5×7 matrices. If rank of A is 3, what is the dimension of $N(A)$? If the dimension of $N(B)$ is 5, what is the rank of B? (Explain each answer to get full credit.)

$r_A + n_A = 7$ by the rank-nullity Theorem; so $r_A = 3 \Rightarrow n_A = 4$

$r_B + n_B = 7$, — " — " —— " — ; so $n_B = 5 \Rightarrow r_B = 2$

b) Let $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ be linearly independent vectors in $\mathbb{R}^5$, and set $\mathbf{v}_1 = \mathbf{u}_3 - 2\mathbf{u}_1$, $\mathbf{v}_2 = \mathbf{u}_1 - 2\mathbf{u}_2$, and $\mathbf{v}_3 = \mathbf{u}_2 - 2\mathbf{u}_3$. Are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ linearly independent? (Explain your answer or get no credit.)

Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha v_1 + \beta v_2 + \gamma v_3 = 0_{\mathbb{R}^5}$. Do we have $\alpha = 0, \beta = 0, \gamma = 0$?

$\alpha v_1 + \beta v_2 + \gamma v_3 = 0_{\mathbb{R}^5} \Leftrightarrow \alpha(u_3 - 2u_1) + \beta(u_1 - 2u_2) + \gamma(u_2 - 2u_3) = 0_{\mathbb{R}^5}$

$\Leftrightarrow (-2\alpha + \beta)u_1 + (-2\beta + \gamma)u_2 + (\alpha - 2\gamma)u_3 = 0_{\mathbb{R}^5}$

$\Leftrightarrow -2\alpha + \beta = 0$ and $-2\beta + \gamma = 0$ and $\alpha - 2\gamma = 0$, since u_1, u_2, u_3 are linearly independent

So $\beta = 2\alpha$, $\gamma = 2\beta$, $\alpha = 2\gamma$; hence $\alpha = 8\alpha$; So $\alpha = 0$, $\beta = 0$, $\gamma = 0$

Hence v_1, v_2 and v_3 are linearly independent.

c) Complete the sentence: The vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ form a basis for $\mathbb{R}^n$ when

i) $\mathbb{R}^n = \operatorname{Span}(u_1, u_2, \ldots, u_n)$

ii) $u_1, u_2, \ldots, u_n$ are linearly independent.

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7. [10, Bonus] Let A be an $n \times n$ matrix. Let the column space of A be denoted $R(A) = \{Ax; x \in \mathbb{R}^n\}$. Show that $N(A) = N(A^2)$ if and only if $R(A) \cap N(A) = \{\mathbf{0}\}$.

Prove

i) $(N(A) = N(A^2)) \Rightarrow R(A) \cap N(A) = \{0_{\mathbb{R}^n}\}$

Let $x \in R(A) \cap N(A)$. Show $x = 0_{\mathbb{R}^n}$

$x \in R(A) \cap N(A) \Rightarrow x = Az$ for some $z \in \mathbb{R}^n$ and $Ax = 0_{\mathbb{R}^n}$

$\Rightarrow A^2 z = Ax = 0_{\mathbb{R}^n}$

$\Rightarrow z \in N(A^2) = N(A)$

$\Rightarrow z \in N(A)$

$\Rightarrow Az = 0_{\mathbb{R}^n}$; so $x = 0_{\mathbb{R}^n}$, and $R(A) \cap N(A) = \{0_{\mathbb{R}^n}\}$

ii) $(R(A) \cap N(A) = \{0_{\mathbb{R}^n}\}) \Rightarrow N(A) = N(A^2)$

Show $N(A) \subseteq N(A^2)$ and $N(A^2) \subseteq N(A)$.

Let $x \in N(A)$, then $Ax = 0_{\mathbb{R}^n}$; so $A(Ax) = 0_{\mathbb{R}^n}$; hence $A^2 x = 0_{\mathbb{R}^n}$ and $x \in N(A^2)$. Therefore $N(A) \subseteq N(A^2)$.

Let $x \in N(A^2)$. Then $A^2 x = 0_{\mathbb{R}^n}$ or $A(Ax) = 0_{\mathbb{R}^n}$; so $Ax \in R(A)$, and $Ax \in N(A)$. Therefore $Ax \in R(A) \cap N(A) = \{0_{\mathbb{R}^n}\}$

Hence $Ax = 0_{\mathbb{R}^n}$, and $x \in N(A)$. Hence $N(A^2) \subseteq N(A)$.

Consequently $N(A) = N(A^2)$ if $R(A) \cap N(A) = \{0_{\mathbb{R}^n}\}$.