MAS 3105 (Linear Algebra)
Test 2, Friday June 05, 2015 — Answers

Name: **PID:**

Remember that you won't get any credit if you do not show the steps to your answers. Total=105 points.

1. [20] Find a basis for the null space and a basis for the column space of the matrix $A = \begin{pmatrix} -1 & 2 & -3 & 5 \\ 2 & 3 & 7 & 1 \\ 6 & 16 & 22 & 14 \end{pmatrix}$.

- Solve $Ax = 0_{\mathbb{R}^3}$ to find $N(A)$, then a basis.

$$\left(\begin{array}{cccc|c} -1 & 2 & -3 & 5 & 0 \\ 2 & 3 & 7 & 1 & 0 \\ 6 & 16 & 22 & 14 & 0 \end{array}\right) \xrightarrow[6r_1+r_3]{2r_1+r_2} \left(\begin{array}{cccc|c} -1 & 2 & -3 & 5 & 0 \\ 0 & 7 & 1 & 11 & 0 \\ 0 & 28 & 4 & 44 & 0 \end{array}\right) \xrightarrow{-4r_2+r_3} \left(\begin{array}{cccc|c} -1 & 2 & -3 & 5 & 0 \\ 0 & 7 & 1 & 11 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$7x_2 + x_3 + 11x_4 = 0 \longrightarrow x_2 = -\frac{x_3}{7} - \frac{11}{7}x_4$$

$$-x_1 + 2x_2 - 3x_3 + 5x_4 = 0 \longrightarrow x_1 = 2x_2 - 3x_3 + 5x_4 = \left(-\frac{2}{7} - 3\right)x_3 + \left(-\frac{22}{7} + 5\right)x_4$$
$$= -\frac{23}{7}x_3 + \frac{13}{7}x_4 \text{ ; set } x_3 = 7\alpha,\ x_4 = 7\beta$$

So $N(A) = \{\alpha(-23, -1, 7, 0)^T + \beta(13, -11, 0, 7)^T ; \alpha, \beta \in \mathbb{R}\}$

Hence $\{(-23, -1, 7, 0)^T, (13, -11, 0, 7)^T\}$ is a basis of $N(A)$.

- The lead variables in the reduction process are x_1 and x_2; so $\left\{ \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 16 \end{pmatrix} \right\}$ is a basis for the column space of A.

---

2. [20] State whether each of the following statement is true or false. No explanations needed.

a) In $\mathbb{R}^3$, there exist four linearly independent vectors. False, $\dim \mathbb{R}^3 = 3$; so you cannot have more than 3 linearly independent vectors

b) If $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are linearly independent in $\mathbb{R}^5$, then they span $\mathbb{R}^5$. False, by Theorem 3.3.4 (i)

c) If A is a 7×5 matrix , then A and A^T have the same rank. True, by Theorem 3.6.6

d) If U is the reduced row echelon form of a nonsingular matrix A, then A and U have the same column space. True as $U = I_n$ if $A \in M_n$ and A nonsingular

e) If If $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ span a subspace of $\mathbb{R}^4$, then they are linearly independent. False; you may have $\text{Span}(x_1, x_2) = \text{Span}(x_1, x_2, x_3, x_4)$

f) If A is a 12×12 nonsingular matrix , then A and A^T have the same nullity. True, $N(A) = \{0_{\mathbb{R}^{12}}\} = N(A^T)$, since $\det(A^T) = \det A \neq 0$

g) If L is a linear operator on $\mathbb{R}^n$ with $\ker(L) = \{0_{\mathbb{R}^n}\}$, then $R(L) = \mathbb{R}^n$. True

h) If U and V are subspaces of a vector space E, then $U + V$ is a subspace of E too. True, see Pb 25 in 3.2

i) If A and B are similar matrices and A is singular, then B is also singular. True. $A = P^{-1}BP$, for some nonsingular matrix P; so $\det A = \det(P^{-1})\det(B)\det(P) = \det(B)$ since $\det(P^{-1}) = \frac{1}{\det P}$

j) If A and B are similar matrices, then A^T and B^T are also similar matrices. True

$A = P^{-1}BP$ for some nonsingular matrix P

$A^T = (P^{-1}BP)^T = P^T B^T (P^{-1})^T = P^T B^T (P^T)^{-1}$

Hence A^T and B^T are similar.

3. [10] Let $\mathcal{M}_2 = \left\{ A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}; a,\ b,\ c,\ d \in \mathbb{R} \right\}$ be the space of 2×2 matrices. Define on $\mathcal{M}_2$ a mapping L by $L(A) = A + A^T$. a) Show that L is linear. b) Find a basis for $\ker(L)$ and a basis for $\mathrm{R}(L)$.

a) Let $A, B \in \mathcal{M}_2$ and $\alpha, \beta \in \mathbb{R}$.

$L(\alpha A + \beta B) = \alpha A + \beta B + (\alpha A + \beta B)^T = \alpha A + \beta B + \alpha A^T + \beta B^T$

$= \alpha A + \alpha A^T + \beta B + \beta B^T = \alpha(A + A^T) + \beta(B + B^T) = \alpha L(A) + \beta L(B)$.

So L is linear.

b) $\ker L = \{A \in \mathcal{M}_2;\ A + A^T = 0_{\mathcal{M}_2}\}$ Let $A \in \ker L$ $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = -A^T = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ so $a = -a \to a = 0$, $d = -d \to d = 0$, $b = -c$

So $A = b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; hence $\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$ = basis of $\ker L$

$R(L) = \{A + A^T; A \in \mathcal{M}_2\} = \left\{\begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}; a, b, c, d \in \mathbb{R}\right\}$

$L(A) = 2a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (b+c)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; Hence $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$ = bases of $R(L)$

---

4. [25] Let $\mathbf{v}_1 = (-1,-2,1)^T$, $\mathbf{v}_2 = (1,3,2)^T$, $\mathbf{v}_3 = (1,1,2)^T$, and $\mathbf{w}_1 = (-1,-3,1)^T$, $\mathbf{w}_2 = (2,3,1)^T$ and $\mathbf{w}_3 = (1,1,3)^T$ be vectors in $\mathbb{R}^3$. Let L be the linear operator defined on $\mathbb{R}^3$ by

$L(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) = (-x_1 + x_2 + 2x_3)\mathbf{v}_1 + (x_1 + 2x_2 - x_3)\mathbf{v}_2 + (2x_1 - x_2 + x_3)\mathbf{v}_3$.

a) Find the matrix representation M of L relative to the ordered basis $B = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. b) Find the transition matrix T from the ordered basis B to the ordered basis $D = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]$. c) Write down the matrix P of L with respect to the ordered basis D in terms of M, but do not attempt to find the entries of P. d) If $\mathbf{u} = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$, find the coordinates of $L(\mathbf{u})$ in the ordered basis D.

a) $L(v_1) = -v_1 + v_2 + 2v_3$, $L(v_2) = v_1 + 2v_2 - v_3$, $L(v_3) = 2v_1 - v_2 + v_3$

Hence $M = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{pmatrix}$

b) $T = T_{B \to D}$. Set $V = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, $W = \begin{pmatrix} -1 & 2 & 1 \\ -3 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$. $T = W^{-1}V$

To find T, find the RREF of the augmented matrix $(W|V)$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 1 & -1 & 1 & 1 \\ -3 & 3 & 1 & -2 & 3 & 1 \\ 1 & 1 & 3 & 1 & 2 & 2 \end{array}\right) \xrightarrow[r_1 + r_3]{-3r_1 + r_2} \left(\begin{array}{ccc|ccc} -1 & 2 & 1 & -1 & 1 & 1 \\ 0 & -3 & -2 & 1 & 0 & -2 \\ 0 & 3 & 4 & 0 & 3 & 3 \end{array}\right) \xrightarrow{r_2 + r_3}$$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 1 & -1 & 1 & 1 \\ 0 & -3 & -2 & 1 & 0 & -2 \\ 0 & 0 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[r_3/2]{\frac{2}{3}r_2 + r_1} \left(\begin{array}{ccc|ccc} -1 & 0 & -1/3 & -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & -3 & -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right) \xrightarrow[2r_3 + r_2]{\frac{1}{3}r_3 + r_1}$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -\frac{1}{6} & \frac{3}{2} & -\frac{1}{6} \\ 0 & -3 & 0 & 2 & 3 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right) \xrightarrow[-r_1]{-r_2/3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{3}{2} & \frac{1}{6} \\ 0 & 1 & 0 & -\frac{2}{3} & -1 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right)$$

Hence $T = \begin{pmatrix} \frac{1}{6} & -\frac{3}{2} & \frac{1}{6} \\ -\frac{2}{3} & -1 & \frac{1}{3} \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$

c) $P = T_{D \to B}^{-1} M T_{D \to B} = TMT^{-1}$

d) $[L(u)]_D = T[L(u)]_B$; $[u]_B = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$[L(u)]_B = M[u]_B = \begin{pmatrix} -3 \\ 8 \\ 3 \end{pmatrix}$. Hence $[L(u)]_D = \begin{pmatrix} -\frac{3}{6} - \frac{24}{2} + \frac{3}{6} \\ \frac{6}{3} - 8 + \frac{3}{3} \\ -\frac{3}{2} + \frac{24}{2} + \frac{3}{2} \end{pmatrix} = \begin{pmatrix} -12 \\ -5 \\ 12 \end{pmatrix}$

5. [15] a) Let A and B be 6×9 matrices. If rank of A is 5, what is the dimension of $N(A)$? If the dimension of $N(B)$ is 6, what is the rank of B? (Explain each answer to get full credit.)

$r_A + n_A = 9$ by RNT; so $n_A = 9 - r_A = 9-5=4$

$r_B + n_B = 9$ by RNT; So $r_B = 9 - n_B = 9-6=3$

b) Use the Wronskian to show that the vectors 1, $e^x - e^{-x}$, $e^x + e^{-x}$ are linearly independent in $C^2([0,1])$.

$$W(1, e^x - e^{-x}, e^x + e^{-x}) = \begin{vmatrix} 1 & e^x - e^{-x} & e^x + e^{-x} \\ 0 & e^x + e^{-x} & e^x - e^{-x} \\ 0 & e^x - e^{-x} & e^x + e^{-x} \end{vmatrix} = (e^x + e^{-x})^2 - (e^x - e^{-x})^2$$
$$= e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})$$
$$= 4 \neq 0 \text{ So}$$

$1, e^x - e^{-x}, e^x + e^{-x}$ are linearly independent in $C^2([0,1])$.

c) Complete the sentence: The vectors $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_n$ form a basis for $\mathbb{R}^n$ when the following two conditions are met:

1) $\mathbb{R}^n = \text{Span}(u_1, u_2, \ldots, u_n)$

2) $u_1, u_2, \ldots, u_n$ are linearly independent

---

6. [10] Let $\mathbf{u}_1 = (-1,1,1)^T$, $\mathbf{u}_2 = (2,a,2)^T$ and $\mathbf{u} = (-1, a^2+2, 5)^T$ be vectors in $\mathbb{R}^3$. For which values of a does the vector $\mathbf{u}$ belong to Span$(\mathbf{u}_1, \mathbf{u}_2)$?

u belongs to Span(u_1, u_2) iff u_1, u_2, u are linearly dependent or $\det(u_1, u_2, u) = 0$.

$$\begin{vmatrix} -1 & 2 & -1 \\ 1 & a & a^2+2 \\ 1 & 2 & 5 \end{vmatrix} = -(5a - 2(a^2+2)) - 2(5 - (a^2+2)) - (2-a)$$
$$= -5a + 2a^2 + 4 - 10 + 2a^2 + 4 - 2 + a$$
$$= 4a^2 - 4a - 4 = 0 \rightarrow a^2 - a - 1 = 0$$

$(a - \frac{1}{2})^2 - \frac{5}{4} = 0 \rightarrow (a - \frac{1}{2})^2 = \frac{5}{4} \rightarrow a = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$. So if $a = \frac{1 \pm \sqrt{5}}{2}$, then u_1, u_2, u are linearly dependent; so $u \in \text{Span}(u_1, u_2)$.

---

7. [5] Let L denote a linear operator on a vector space E. Let U denote a subspace of E. Set $L^{-1}(U) = \{v \in E; L(v) \in U\}$. Show that $L^{-1}(U)$ is a subspace of E.

- Show $L^{-1}(U) \neq \phi$. $0_E \in E$, and $L(0_E) = 0_E \in U$, since U is a subspace of E. So $L^{-1}(U) \neq \phi$ as $0_E \in L^{-1}(U)$.
- Let $u, v \in L^{-1}(U)$ show $u+v \in L^{-1}(U)$. $u \in L^{-1}(U) \Rightarrow L(u) \in U$; $v \in L^{-1}(U) \Rightarrow L(v) \in U$; So $L(u) + L(v) \in U$, as U is a subspace of E. Now $L(u+v) = L(u) + L(v)$, as L is linear; So $L(u+v) \in U$; hence $u+v \in L^{-1}(U)$.
- Let $\alpha \in \mathbb{R}$, let $v \in L^{-1}(U)$. Show $\alpha v \in L^{-1}(U)$. $v \in L^{-1}(U) \Rightarrow L(v) \in U \Rightarrow \alpha L(v) \in U$, as U subspace of E. Now $L(\alpha v) = \alpha L(v)$ as L is linear; So $L(\alpha v) \in U$; hence $\alpha v \in L^{-1}(U)$.

Therefore $L^{-1}(U)$ is a subspace of E.