MAC 2312 (Calculus II) — Answers

Test 3, Wednesday November 23, 2016

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; no credits will be awarded to unexplained answers. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration.

1. [10] Use multiplication to find the first three nonzero terms of the Maclaurin series for $f(x) = e^x \cos(x^2)$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$\cos(x^2) = 1 - \frac{x^4}{2} + \frac{x^8}{24} + \cdots, \quad e^x\cos(x^2) = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)\left(1-\frac{x^4}{2}+\frac{x^8}{24}+\cdots\right)$$

$$e^x\cos(x^2) = 1 + x + \frac{x^2}{2} + \cdots$$

2. [10] Find the Taylor polynomial of order four for $f(x) = \sin(x/2)$ about $x = \pi$.

$$f'(x) = \frac{1}{2}\cos(x/2), \quad f''(x) = -\frac{1}{4}\sin(x/2), \quad f^{(3)}(x) = -\frac{1}{8}\cos(x/2)$$

$$f^{(4)}(x) = \frac{1}{16}\sin(x/2)$$

$$f(\pi) = 1, \quad f'(\pi) = 0, \quad f''(\pi) = -\frac{1}{4}, \quad f^{(3)}(\pi) = 0, \quad f^{(4)}(\pi) = \frac{1}{16}$$

$$P_4(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2}(x-\pi)^2 + \frac{f^{(3)}(\pi)}{6}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{24}(x-\pi)^4$$

$$= 1 - \frac{(x-\pi)^2}{8} + \frac{(x-\pi)^4}{16(24)}$$

7. [10] Determine the radius of convergence and the interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(-1)^k (x-2)^k}{k4^k}$.

$$\rho = \lim_{k\to\infty} \frac{|(x-2)^{k+1}|}{(k+1)4^{k+1}} \cdot \frac{k4^k}{|x-2|^k} = \frac{|x-2|}{4} \lim_{k\to\infty} \frac{k}{k+1} = \frac{|x-2|}{4} < 1 \to |x-2| < 4$$

So $R = 4$. $\quad -4 < x-2 < 4 \to -2 < x < 6$

At $x = -2$: $\sum_{k=1}^{\infty} \frac{(-1)^k(-4)^k}{k4^k} = \sum_{k=1}^{\infty} \frac{4^k}{k4^k} = \sum_{k=1}^{\infty} \frac{1}{k}$, diverges; harmonic series

At $x = 6$: $\sum_{k=1}^{\infty} \frac{(-1)^k 4^k}{k4^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, converges; A.S.T.

So $I_c = (-2, 6]$.

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8. [8] Show that the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}$ satisfies the requirements of the alternating series test. Find a value of n for which the n^{th} partial sum is ensured to approximate the series to within three-decimal place accuracy.

Set $S_n = \sum_{k=1}^{n} \frac{(-1)^k}{k^4}$, $S = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}$

$|S_n - S| \le \frac{1}{(n+1)^4}$. $|S_n - S| \le 5(10^{-4})$ if $\frac{1}{(n+1)^4} \le 5(10^{-4}) \to \frac{10^4}{5} \le (n+1)^4$

$\to \sqrt[4]{\frac{10^4}{5}} \le n+1 \to \frac{10}{\sqrt[4]{5}} - 1 \le n.$

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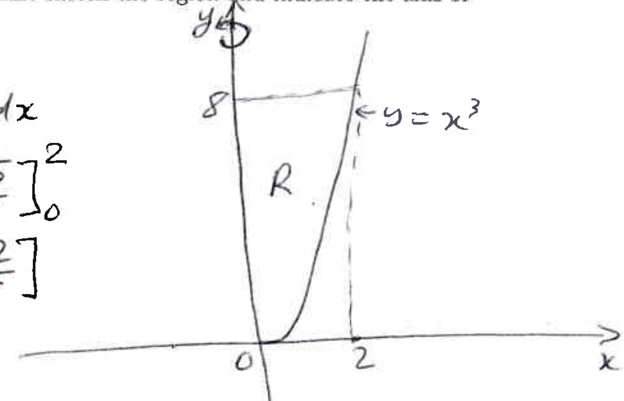
9. [8] Use cylindrical shells to find the volume of the solid that results when the region enclosed by the curves $x = 0$, $y = x^3$, and $y = 8$, is revolved about the y-axis. You must sketch the region and indicate the axis of revolution.

$$V = \int_0^{\cancel{8}\,2} 2\pi x(8 - x^3)\,dx = \int_0^2 2\pi x(8 - x^3)\,dx$$

$$= 2\pi \left[4x^2 - \frac{x^5}{5}\right]_0^2 = 2\pi\left[4x^2 - \frac{x^5}{5}\right]_0^2$$

$$= 2\pi\left(16 - \frac{32}{5}\right) = 2\pi\left[16 - \frac{32}{5}\right] =$$

$$= \frac{96\pi}{5}$$



3. [10] Find the area of the surface generated by revolving the curve $y = x^3$, $0 \le x \le 1$ about the x-axis.

$$S = 2\pi\int_0^1 x^3\sqrt{1+(3x^2)^2}\,dx = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx; \quad u = 1+9x^4;\ du = 36x^3dx$$
$$= \frac{2\pi}{36}\int_1^{10}\sqrt{u}\,du = \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{\pi}{27}(10\sqrt{10}-1)$$

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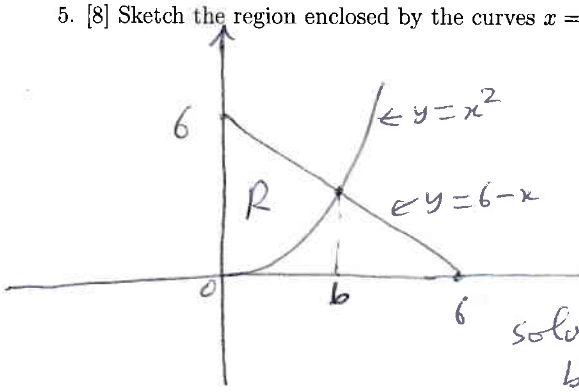
4. [10] a) Use a popular Maclaurin series to find the Maclaurin series for $f(x) = \frac{1}{1-x^2}$, and specify its interval of convergence. b) Find the derivative function f' of f, and use the Maclaurin series obtained in part a) and a well-known theorem to write down the Maclaurin series for f'. c) Use the result in b) to derive the sum of the series $\sum_{k=1}^{\infty}\frac{k}{2^{2k-2}}$.

a) $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, so $\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k}$, $I_c = (-1, 1)$

b) $f'(x) = \frac{2x}{(1-x^2)^2} = \frac{d}{dx}\sum_{k=0}^{\infty} x^{2k} \overset{\text{T.T.D}}{=} \sum_{k=1}^{\infty}\frac{d}{dx}(x^{2k}) = \sum_{k=1}^{\infty} 2k\,x^{2k-1}$

c) For $x = \frac{1}{2}$, $\sum_{k=1}^{\infty} 2k\left(\frac{1}{2}\right)^{2k-1} = \sum_{k=1}\frac{2k}{2^{2k-1}} = \sum_{k=1}\frac{k}{2^{2k-2}} = f'(1/2) = \frac{1}{\left(\frac{3}{4}\right)^2} = \frac{16}{9}$

5. [8] Sketch the region enclosed by the curves $x = 0$, $y = x^2$, $x + y = 6$, and find its area.



$$A = \int_0^2 6-x-x^2\,dx = 6x - \frac{x^2}{2} - \frac{x^3}{3}\Big]_0^2$$
$$= 6(2) - \frac{4}{2} - \frac{8}{3}$$
$$= 10 - \frac{8}{3}$$
$$= \frac{22}{3}$$

solve $b^2 = 6-b$
$b^2+b-6=0$
$(b+3)(b-2)=0$
$b=2$

6. [12] a) Find the exact length of the arc of the parametric curve $x = t^2$, $y = t^3$, $1 \le t \le 2$.

$x'(t) = 2t$, $y'(t) = 3t^2$

$$L = \int_1^2\sqrt{x'(t)^2+y'(t)^2}\,dt$$
$$= \int_1^2\sqrt{4t^2+9t^4}\,dt = \int_1^2 t\sqrt{4+9t^2}\,dt = \frac{1}{18}\int_{13}^{40}\sqrt{u}\,du = \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big]_{13}^{40}$$
$$= \frac{1}{27}(40\sqrt{40}-13\sqrt{13})$$

$u = 4+9t^2$
$du = 18t\,dt$

b) Find the volume of the solid that results when the region enclosed by the curves $x = 0$, $y = x^2$, and $y = 6 - x$ is revolved about the x-axis.

See Pb 5

$$V = \pi\int_0^2 (6-x)^2 - x^4\,dx = \pi\left[-\frac{(6-x)^3}{3} - \frac{x^5}{5}\right]_0^2$$
$$= \pi\left[\frac{-4^3+6^3}{3} - \frac{2^5}{5}\right]$$