MAC 2313 (Calculus III) — Answers
Test 3 Thursday July 06, 2017

Name: **PID:**

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work. Guessing the correct answers won't give you any credits. You must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may use the back of the page to show your work. 3 pages. Total=90 points. Always do your best.

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1. [15] a) Find the point Q on the sphere $x^2+y^2+z^2=5$ that is closest to the point $P(-2,0,-4)$. b) Find the distance between Q and P.

Introduce the distance function $f(x,y,z)=(x+2)^2+y^2+(z+4)^2$

Set $g(x,y,z)=x^2+y^2+z^2-5$. $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$

$\langle 2(x+2), 2y, 2(z+4)\rangle = \lambda\langle 2x, 2y, 2z\rangle$; so

$(1-\lambda)x=-2$, $(1-\lambda)y=0$, $(1-\lambda)z=-4$. Thus $1-\lambda\neq 0$; so $y=0$

$x=-\frac{2}{1-\lambda}=\left(\frac{-4}{1-\lambda}\right)\left(\frac{1}{2}\right)=\frac{z}{2}$, or $z=2x$; $x^2+4x^2=5$, so $x^2=1$.

Hence $x=\pm 1$. If $x=1$, then $z=2$; $f(1,0,2)=3^2+0+6^2=45$

If $x=-1$, then $z=-2$; $f(-1,0,-2)=1^2+0+2^2=5$

Hence $Q=(-1,0,-2)$ and $d(P,Q)=\sqrt{5}$

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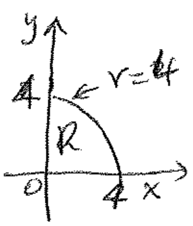
2. [20] a) Use spherical coordinates to find the volume of the solid G in the first octant that is bounded above by the cone $z=8-\sqrt{x^2+y^2}$, and below by the plane $z=4$. Just set up the triple integral including all integration limits, but do not evaluate it.



$4\le \rho\cos\phi\le 8-\rho\sin\phi$ and $4\le 8-\rho\sin\phi$

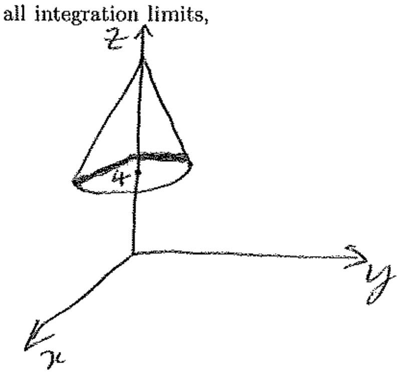
$4\sec\phi\le\rho\le\frac{8}{\cos\phi+\sin\phi}$

So $\rho\le 4\csc\phi$; thus $\frac{4}{\cos\phi}\le\frac{4}{\sin\phi}$; so $\sin\phi\le\cos\phi$ or

$\tan\phi\le 1$; hence $0\le\phi\le\frac{\pi}{4}$

At intersection, $8-r=4$ or $r=4$.

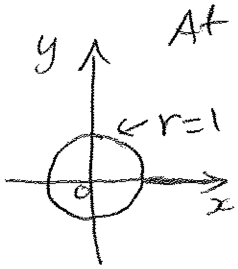
$$V=\int_0^{\pi/2}\int_0^{\pi/4}\int_{4\sec\phi}^{\frac{8}{\cos\phi+\sin\phi}}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$



b) Use cylindrical coordinates to evaluate the volume of the solid G that lies inside the paraboloid $z=x^2+y^2$, but not above the cone $z=\sqrt{x^2+y^2}$.

At intersection, $r^2=r$ or

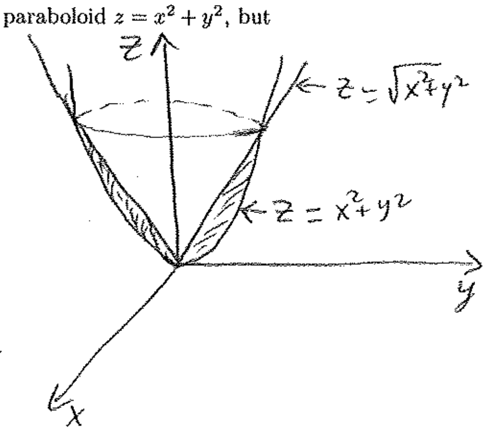
$r(r-1)=0$; so $0\le r\le 1$



$$V=\int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta$$

$$=2\pi\int_0^1 r(r-r^2)\,dr$$

$$=2\pi\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}$$

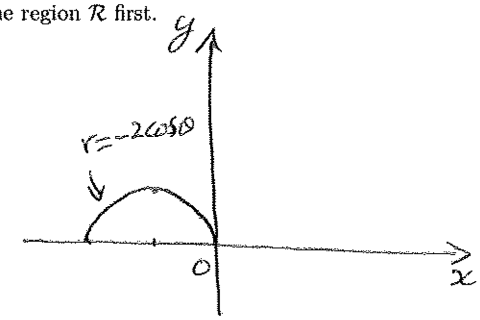


3. [10] Express the integral $\iint_{\mathcal{R}} \cos((x^2+y^2)^{\frac{3}{2}})\,dA$ in polar coordinates, but do not evaluate it. $\mathcal{R}$ is the region that lies in the second quadrant and is enclosed by the curve $x^2+y^2=-2x$. You must sketch the region $\mathcal{R}$ first.

$x^2+y^2=-2x \rightarrow r^2=-2r\cos\theta \rightarrow r=-2\cos\theta$

$(x+1)^2+y^2=1$

$$\iint_R \cos((x^2+y^2)^{3/2})\,dA = \int_{\pi/2}^{\pi}\int_0^{-2\cos\theta} r\cos(r^3)\,dr\,d\theta$$



4. [12] Evaluate each integral [12]

a) $\int_1^2\int_0^{\ln(y)} ye^{2x}\,dx\,dy =$

$$\int_1^2 y\left[\frac{e^{2x}}{2}\right]_0^{\ln(y)} dy$$

$$= \int_1^2 y\,\frac{(e^{2\ln(y)}-1)}{2}\,dy;\quad e^{2\ln y}=e^{\ln y^2}=y^2$$

$$= \frac{1}{2}\int_1^2 y(y^2-1)\,dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4}-\frac{y^2}{2}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{16-1}{4}-\left(4-\frac{1}{2}\right)\right]$$

b) $\int_1^3\int_y^3\int_0^x \frac{x}{x^2+z^2}\,dz\,dx\,dy = \int_1^3\int_y^3\int_0^1 \frac{x^2}{x^2(1+u^2)}\,du\,dx\,dy$

Set $z=xu$
$dz=x\,du$

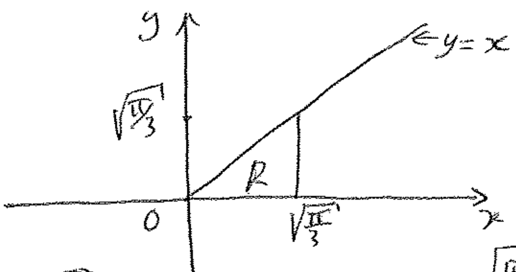
$$= \int_1^3 (3-y)\left[\arctan u\right]_0^1 dy$$

$$= \int_1^3 (3-y)\,dy\ \frac{\pi}{4}$$

$$= \frac{\pi}{4}\left[-\frac{(3-y)^2}{2}\right]_1^3$$

$$= \frac{\pi}{4}\left(0+\frac{4}{2}\right) = \frac{\pi}{2}$$

5. [8] Use an appropriate order of integration to evaluate the double integral $\int_0^{\sqrt{\frac{\pi}{3}}}\int_y^{\sqrt{\frac{\pi}{3}}} \sin(x^2)\,dx\,dy$.



$$\int_0^{\sqrt{\pi/3}}\int_0^x \sin(x^2)\,dy\,dx = \int_0^{\sqrt{\pi/3}} x\sin(x^2)\,dx = -\frac{\cos(x^2)}{2}\Big]_0^{\sqrt{\pi/3}}$$

$$= \frac{1}{2}\left[-\cos(\pi/3)+\cos(0)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}+1\right] = \frac{1}{4}$$

6. [13] Let $f(x,y) = 8x^3 - 24xy + y^3 + 7$. Find all the critical points of f and classify each of them as a local maximum, a local minimum, or a saddle point.

$f_x(x,y) = 24x^2 - 24y \qquad f_y(x,y) = -24x + 3y^2$

$f_x(x,y) = 0 \rightarrow y = x^2, \quad f_y(x,y) = 0 \rightarrow 8x = y^2 = (x^2)^2 = x^4$

$8x = x^4 \rightarrow x(8 - x^3) = 0 \rightarrow x = 0$ or $x^3 = 8 \rightarrow x = 2$

C.P.s: $(0,0)$, $(2,4)$

$f_{xx}(x,y) = 48x, \quad f_{xy}(x,y) = -24, \quad f_{yy}(x,y) = 6y$

$D(0,0) = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}(0,0)^2 = 0 - 24^2 < 0$; f has a saddle point at $(0,0)$

$D(2,4) = f_{xx}(2,4) f_{yy}(2,4) - f_{xy}(2,4)^2 = 48(2)6(4) - 24^2 = 48^2 - 24^2 > 0$

$f_{xx}(2,4) = 48(2) > 0$; So f has a local minimum at $(2,4)$.

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7. [12] Set $u = e^{2x} - y$ and $v = e^{2x} + y$. a) Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ and express it in terms of u and v. b) Use an appropriate change of variables to evaluate the integral $\displaystyle\iint_{\mathcal{R}} \frac{e^{2x}}{e^{2x} + y}\, dA$, where $\mathcal{R}$ is the region enclosed by the curves $y = e^{2x}$, $y = e^{2x} - 2$, $y = 1 - e^{2x}$ and $y = 4 - e^{2x}$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2e^{2x} & -1 \\ 2e^{2x} & 1 \end{vmatrix} = 2e^{2x} + 2e^{2x} = 4e^{2x}$$

$\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{4e^{2x}}$; Now $u + v = 2e^{2x}$; so $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2(u+v)}$

b) Set $u = e^{2x} - y$, $v = e^{2x} + y$; then $0 \le u \le 2$ and $1 \le v \le 4$

Hence

$$\iint_R \frac{e^{2x}}{e^{2x}+y}\, dA = \int_0^2 \int_1^4 \frac{(u+v)/2}{v} \cdot \frac{1}{2(u+v)}\, dv\, du$$

$\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$

$$= \frac{1}{4} \int_0^2 \ln v \Big]_1^4 du$$

$$= \frac{2}{4}(\ln 4 - \ln 1)$$

$$= \frac{1}{2}(2\ln 2) = \ln 2, \text{ as } \ln 1 = 0.$$