MAC 2313 (Calculus III) — key
Test 3, Wednesday March 25, 2015

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. *You will not get any credit to any of the problems if you do not show your work.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Total=65 points. Good luck.**

1. [10] a) Find the point Q on the plane $2x - 3y + z = 14$ that is closest to the point $B(-1, 1, -2)$. b) Find the distance between Q and B.

Let $Q(x,y,z)$ lie on plane. $d(B,Q) = \sqrt{(x+1)^2 + (y-1)^2 + (z+2)^2}$

a) Set $f(x,y,z) = (x+1)^2 + (y-1)^2 + (z+2)^2$ and $g(x,y,z) = 2x - 3y + z - 14$

We have $\nabla f(x,y,z) = 2\langle x+1, y-1, z+2\rangle = \lambda \nabla g(x,y,z) = \lambda\langle 2, -3, 1\rangle$

So $x = -1 + \lambda$, $y = 1 - \frac{3\lambda}{2}$, $z = -2 + \frac{\lambda}{2}$; reporting this in the plane equation, we find $2(-1+\lambda) - 3(1 - \frac{3\lambda}{2}) + (-2 + \frac{\lambda}{2}) = 14$ or

$-7 + 7\lambda = 14$; so $7\lambda = 21$ or $\lambda = 3$; hence $Q(2, -\frac{7}{2}, -\frac{1}{2})$

b) $d(B,Q) = \sqrt{3^2 + (\frac{9}{2})^2 + (\frac{3}{2})^2}$

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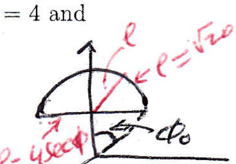
2. [20] a) Use spherical coordinates to express the volume of the solid G bounded below by the plane $z = 4$ and above by the sphere $x^2 + y^2 + z^2 = 20$. Be sure to include all integration limits, but do not evaluate.



$z = 4 \rightarrow \rho\cos\phi = 4 \rightarrow \rho = 4\sec\phi$

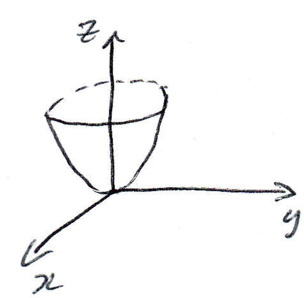
$x^2 + y^2 + 16 = 20 \rightarrow \rho^2\sin^2\phi = 4$ at intersection

$\rho\cos\phi_0 = 4$ and $\rho\sin\phi_0 = 2$; so $\tan\phi_0 = \frac{2}{4} = \frac{1}{2}$

$\phi_0 = \tan^{-1}(1/2)$; Solid is contained in the cone

$\phi = \phi_0$; so $V = \int_0^{2\pi}\int_0^{\phi_0}\int_{4\sec\phi}^{2\sqrt{5}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

b) Use cylindrical coordinates to find the volume of the solid G bounded above by the plane $z = 4$ and below by the paraboloid $z = x^2 + y^2$.



At intersection $x^2 + y^2 = 4$

$r^2 = 4$

so $0 \leq r = 2$

$0 \leq \theta \leq 2\pi$

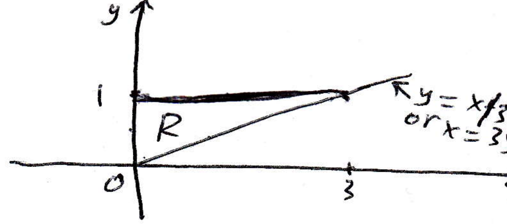
$$V = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^2 r(4 - r^2)\,dr$$

$$= 2\pi \left[-\frac{(4-r^2)^2}{4}\right]_0^2 = \frac{2\pi}{4}(0 + 16) = 8\pi$$

3. [8] Use an appropriate order of integration to evaluate the double integral $\int\int_{\mathcal{R}} \sin(y^2)\,dA$, where $\mathcal{R}$ is the region in the first quadrant enclosed by $y = 1$, $x = 0$, $x = 3$ and $y = x/3$. You must sketch the region $\mathcal{R}$ first.

$$\iint_R \sin(y^2)\,dA = \int_0^1\int_0^{3y} \sin(y^2)\,dx\,dy$$
$$= \int_0^1 3y\sin(y^2)\,dy$$
$$= -\frac{3}{2}\cos(y^2)\Big]_0^1$$
$$= \frac{3}{2}(1-\cos(1))$$



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4. Evaluate each integral [12]

a) $\displaystyle\int_0^\pi\int_0^x \frac{\sin x}{x}\,dydx =$ $\displaystyle\int_0^\pi \cancel{x}\,\frac{\sin x}{\cancel{x}}\,dx$

$$= -\cos x\Big]_0^\pi$$
$$= 1+1$$
$$= 2$$

b) $\displaystyle\int_1^2\int_y^2\int_0^x \frac{x}{x^2+z^2}\,dzdxdy =$ $\displaystyle\int_1^2\int_y^2\int_0^1 \frac{x^2\,du}{x^2(1+u^2)}\,dx\,dy$

$z = xu$
$dz = x\,du$

$$= \int_1^2\int_y^2\int_0^1 \frac{du}{1+u^2}\,dx\,dy$$
$$= \int_1^2\int_y^2 \tan^{-1}u\Big]_0^1 dx\,dy$$
$$= \frac{\pi}{4}\int_1^2 (2-y)\,dy$$
$$= \frac{\pi}{4}\left(-\frac{(2-y)^2}{2}\Big]_1^2\right)$$
$$= \frac{\pi}{8}$$

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5. [15] a) If $u = x^2 - y^2$ and $v = xy$, find the Jacobian $\partial(x, y)/\partial(u, v)$ and express it in terms of u and v.

b) Use an appropriate change of variables to evaluate $\displaystyle\int\int_R \frac{x^4 - y^4}{xy + 1}\,dydx$, where R is the region in the first quadrant enclosed by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 3$, $xy = 2$, $xy = 5$.

a) $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2x^2+2y^2 = 2(x^2+y^2)$$

$$(x^2+y^2)^2 = x^4+2x^2y^2+y^4 = x^4-2x^2y^2+y^4+4x^2y^2 = (x^2-y^2)^2+4(xy)^2$$
$$= u^2+4v^2\;;\; \text{So } x^2+y^2 = \sqrt{u^2+4v^2}$$

Hence $$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2\sqrt{u^2+4v^2}}$$

b) Set $u = x^2-y^2$, $v = xy$; note $x^4-y^4 = (x^2-y^2)(x^2+y^2) = u\sqrt{u^2+4v^2}$

$$\iint_R \frac{x^4-y^4}{xy+1}\,dA = \int_1^3\int_2^5 \frac{u\cancel{\sqrt{u^2+4v^2}}}{v+1}\cdot\frac{1}{2\cancel{\sqrt{u^2+4v^2}}}\,dv\,du$$
$$= \frac{1}{2}\int_1^3 u\,du\int_2^5 \frac{dv}{v+1}$$
$$= \frac{1}{2}\left[\frac{u^2}{2}\right]_1^3\left[\ln(v+1)\right]_2^5$$
$$= \frac{1}{2}\left(\frac{9}{2}-\frac{1}{2}\right)(\ln 6 - \ln 3)$$
$$= 2\ln\frac{6}{3}$$
$$= 2\ln 2$$