MAP 2302 (Differential Equations) — Answers
TEST 3, Friday April 22, 2016

Name: **PID:**

Remember that no documents or calculators are allowed during the test. You must show all your work to deserve the full credit assigned to any question. 4 pages.

1.[10] Use your knowledge of step functions to find the Laplace transform of the function f defined by:

$$f(t) = \begin{cases} 2t, & 0 \le t < 3, \\ 4, & 3 \le t < 4, \\ 2-t, & t \ge 4. \end{cases}$$

$$f(t) = 2t(1-u_3(t)) + 4(u_3(t) - u_4(t)) + (2-t)u_4(t)$$
$$= 2t - 2(t-3)u_3(t) - 2u_3(t) - (t-4)u_4(t) - 6u_4(t)$$

$$\mathcal{L}f(s) = 2\mathcal{L}(t)(s) - 2\mathcal{L}(u_3(t)\cdot(t-3))(s) - 2\mathcal{L}(u_3(t))(s) - \mathcal{L}(u_4(t)\cdot(t-4)) - 6\mathcal{L}(u_4)(s)$$
$$= \frac{2}{s^2} - \frac{2e^{-3s}}{s^2} - \frac{2e^{-3s}}{s} - \frac{e^{-4s}}{s^2} - \frac{6e^{-4s}}{s}, \quad s>0$$

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2. [20] Use Laplace transform to solve the differential equation: $y'' + 9y = \cos(3t)$, $y(0) = 2$, $y'(0) = -6$.

$$\mathcal{L}(y''+9y)(s) = \mathcal{L}(\cos(3t))(s) = \frac{s}{s^2+9}. \quad \text{Set } Y = \mathcal{L}y$$
$$(s^2Y - sy(0) - y'(0) + 9Y)(s) = \frac{s}{s^2+9}$$
$$(s^2+9)Y = 2s - 6 + \frac{s}{s^2+9}; \text{ so } Y = \frac{2s-6}{s^2+9} + \frac{s}{(s^2+9)^2}$$
$$y(t) = \mathcal{L}^{-1}Y(t)$$
$$= 2\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right)(t) - 2\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)(t) + \mathcal{L}^{-1}\left(\frac{s}{(s^2+9)^2}\right)(t)$$
$$= 2\cos(3t) - 2\sin(3t) + \mathcal{L}^{-1}\left(-\frac{1}{6}\frac{d}{ds}\left(\frac{3}{s^2+9}\right)\right)(t)$$
$$= 2\cos(3t) - 2\sin(3t) + \frac{1}{6}t\sin(3t)$$

3. [20] Use the method of Frobenius to find two linearly independent series solutions of the form $x^r \sum_{n=0}^{\infty} c_n x^n$ to the differential equation: $2x^2y'' - 3xy' + (2x^2+2)y = 0, \quad 0 < x < R$. Include the first three terms of each series solution.

Seek $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $c_0 \neq 0$. So $y' = \sum_{n=0}^{\infty} c_n x^{n+r-1}(n+r)$

$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$. So D.E becomes:

$$2\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} - 3\sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + 2\sum_{n=0}^{\infty} c_n x^{n+r+2} + 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0,$$ for all x in $(0, R)$

Now $2\sum_{n=0}^{\infty} c_n x^{n+r+2} = 2\sum_{n=2}^{\infty} c_{n-2} x^{n+r}$, by replacing n with $n-2$

Consequently, we have now:

$$\sum_{n=0}^{\infty} \left(2(n+r)(n+r-1) - 3(n+r) + 2\right) c_n x^{n+r} + 2\sum_{n=2}^{\infty} c_{n-2} x^{n+r} = 0$$

Pulling out the first two terms from the first sum:

$$(2r(r-1) - 3r + 2)c_0 x^r + (2(r+1)r - 3(r+1) + 2)c_1 x^{r+1} + \sum_{n=2}^{\infty} \left[\left((n+r)(2(n+r)-5) + 2\right)c_n + 2c_{n-2}\right] x^{n+r} = 0;$$ all coefficients are zero,

<u>Indicial eqn</u> $(2r^2 - 5r + 2)c_0 = 0$, $c_0 \neq 0 \to (2r-1)(r-2) = 0$

$r_1 = 2$, $r_2 = 1/2$

For $r = 2$: $(2(3)(2) - 3(3) + 2)c_1 = 0 \to 5c_1 = 0 \to c_1 = 0$

$((n+2)(2n-1) + 2)c_n + 2c_{n-2} = 0$, $n = 2, 3, 4, \ldots$

$n=2$: $14c_2 + 2c_0 = 0 \to c_2 = -c_0/7$

$n=3$: $27c_3 + 2c_1 = 0 \to c_3 = 0, \to c_5 = 0, \ldots, c_{2p+1} = 0$, $p \geq 0$

$n=4$: $44c_4 + 2c_2 = 0 \to c_4 = -c_2/22 = \dfrac{c_0}{154}$

$$y_1 = x^2\left(1 - \frac{x^2}{7} + \frac{x^4}{154} + \cdots\right),$$ setting $c_0 = 1$

For $r = 1/2$: $\left(\frac{3}{2} - 3\left(\frac{3}{2}\right) + 2\right)c_1 = 0 \to c_1 = 0$

$((n+\frac{1}{2})(2n-4) + 2)c_n + 2c_{n-2} = 0$, $n = 2, 3, 4, \ldots$

$n=2$: $2c_2 + 2c_0 = 0 \to c_2 = -c_0$

$n=3$: $9c_3 + 2c_1 = 0 \to c_3 = 0 \to c_5 = 0 \to \cdots c_{2p+1} = 0$, $p \geq 0$

$n=4$: $\left(\frac{9}{2}(4) + 2\right)c_4 + 2c_2 = 0 \to 20c_4 + 2c_2 = 0 \to c_4 = -c_2/10 = c_0/10$

$$y_2 = x^{1/2}\left(1 - x^2 + \frac{x^4}{10} + \cdots\right),$$ setting $c_0 = 1$

4. [15] Find power series solutions in powers of x, including powers of x up to x^5, of the differential equation: $(x^2-2)y''+xy'-4y=0$.

Seek $y=\sum_{n=0}^{\infty} b_n x^n$. So $y'=\sum_{n=1}^{\infty} b_n n x^{n-1}$, $y''=\sum_{n=2}^{\infty} n(n-1) b_n x^{n-2}$

D.E becomes:

$$\sum_{n=0}^{\infty} n(n-1) b_n x^n - 2\sum_{n=2}^{\infty} n(n-1) b_n x^{n-2} + \sum_{n=0}^{\infty} n b_n x^n - 4\sum_{n=0}^{\infty} b_n x^n = 0,$$

for all x close to zero

Replace $n-2$ with n

$$\sum_{n=0}^{\infty} (n(n-1) b_n + n b_n - 4 b_n) x^n - 2\sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} x^n = 0$$

or $$\sum_{n=0}^{\infty} [(n^2-4) b_n - 2(n+2)(n+1) b_{n+2}] x^n = 0$$

Hence

$$(n^2-4) b_n - 2(n+2)(n+1) b_{n+2} = 0, \quad n = 0, 1, 2, 3, \dots$$

$n=0$: $\quad -4b_0 - 4b_2 = 0 \rightarrow b_2 = -b_0$

$n=1$: $\quad -3b_1 - 12b_3 = 0 \rightarrow b_3 = -b_1/4$

$n=2$: $\quad 0b_2 - 24b_4 = 0 \rightarrow b_4 = 0 \rightarrow b_6 = 0 \rightarrow b_{2p} = 0 \quad p \geq 2$

$n=3$: $\quad 5b_3 - 40b_5 = 0 \rightarrow b_5 = b_3/8 = -b_1/32$

Hence

$$y = b_0(1-x^2) + b_1\left(x - \frac{x^3}{4} - \frac{x^5}{32} + \dots\right)$$

Now b_0, b_1 = arbitrary constants

$y_1 = 1-x^2$ is just a quadratic polynomial, since

$b_{2p} = 0, \; p \geq 2$

5. [20] a) Find the inverse Laplace transform of the function $F(s) = \frac{4s-3}{s(s^2+9)} + \frac{(4s-3)e^{-\pi s}}{s(s^2+9)}$.

$$\frac{4s-3}{s(s^2+9)} = \frac{a}{s} + \frac{bs+c}{s^2+9} = \frac{a(s^2+9)+(bs+c)s}{s(s^2+9)}; \quad \text{So}$$

$$a(s^2+9) + (bs^2+cs) = 4s-3.$$

For $s=0$: $9a = -3 \to a = -1/3$

For $s = 3i$: $-9b + 3ic = 12i - 3 \to -9b = -3 \to b = 1/3$

$\to 3c = 12 \to c = 4$

$$\mathcal{L}^{-1}F(t) = \mathcal{L}^{-1}\left(\frac{-1/3}{s}\right)(t) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right)(t) + \frac{4}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)(t)$$

$$+ u_\pi(t)\left(-\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right)(t-\pi) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right)(t-\pi) + \frac{4}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)(t-\pi)\right)$$

$$= -\frac{1}{3} + \frac{1}{3}\cos(3t) + \frac{4}{3}\sin(3t) + u_\pi(t)\left(-\frac{1}{3} + \frac{1}{3}\cos(3(t-\pi)) + \frac{4}{3}\sin(3(t-\pi))\right)$$

$$= \begin{cases} -\frac{1}{3} + \frac{1}{3}\cos(3t) + \frac{4}{3}\sin(3t), & 0 < t < \pi \\ -\frac{2}{3}, & t > \pi \end{cases}$$

b) Use properties of Laplace transform and the convolution to find the inverse Laplace transform of $G(s) = \frac{6s+7}{(s^2+25)^2}$.

Note $\frac{6s}{(s^2+25)^2} = -\frac{6}{10}\frac{d}{ds}\left(\frac{5}{s^2+25}\right) = \frac{3}{5}\mathcal{L}(t\sin(5t))$ (monomial factor property)

$$\frac{7}{(s^2+25)^2} = \frac{7}{25} \cdot \frac{5}{s^2+25} \cdot \frac{5}{s^2+25}$$

$$= \frac{7}{25}\mathcal{L}(\sin(5t) * \sin(5t))(s)$$

$$= \frac{7}{25}\int_0^t \sin(5(t-r))\sin(5r)\,dr$$

$$= \frac{7}{50}\int_0^t \cos(5(t-r)-5r) - \cos(5(t-r)+5r)\,dr$$

$$= \frac{7}{50}\left\{-\left.\frac{\sin(5t-10r)}{10}\right]_0^t - t\cos(5t)\right\}$$

$$= \frac{7}{50}\left\{\frac{\sin(5t)}{5} - t\cos(5t)\right\}$$

Hence

$$\mathcal{L}^{-1}G(t) = \frac{3}{5}t\sin(5t) + \frac{7}{250}\sin(5t) - \frac{7}{50}t\cos(5t)$$