MAP 2302 (Differential Equations) — Answers
Test 3, Friday April 13, 2018

Name:

PID:

Remember that no documents or calculators are allowed during the test. You shall show all your work to deserve the full mark assigned to any question. Always do your best.

$\mathcal{L}(t^n)(s) = n!/s^{n+1}$, $\mathcal{L}(\sin(bt))(s) = b/(s^2+b^2)$, $\mathcal{L}(\cos(bt))(s) = s/(s^2+b^2)$, $\mathcal{L}(e^{at})(s) = 1/s - a$.

1. [20] We denote the inverse Laplace transform by $\mathcal{L}^{-1}$. Find $\mathcal{L}^{-1}(\frac{2s^2+s+8)}{(s^2-s-12)(s^2+4)})$

$$\frac{2s^2+s+8}{(s^2-s-12)(s^2+4)} = \frac{2s^2+s+8}{(s-4)(s+3)(s^2+4)} = \frac{A}{s-4} + \frac{B}{s+3} + \frac{Cs+D}{s^2+4}$$

$$= \frac{A(s+3)(s^2+4) + B(s-4)(s^2+4) + (Cs+D)(s-4)(s+3)}{(s-4)(s+3)(s^2+4)}$$

Hence

$$A(s+3)(s^2+4) + B(s-4)(s^2+4) + (Cs+D)(s+3)(s-4) = 2s^2+s+8$$

For A, set $s=4$: $A(7)(20) = 2(16)+12 = 44 \to A = \frac{44}{20(7)} = \frac{11}{35}$

For B, set $s=-3$: $B(-7)(13) = 2(9)+5 = 23 \to B = -\frac{23}{91}$

For C & D, set $s=2i$: $(2iC+D)(-4-2i-12) = 2(-4)+2i+8 = 2i$

$$(2Ci+D)(-16-2i) = 2i$$

$$(2Ci+D)(8+i) = -i$$

$$-2C+8D+(16C+D)i = -i$$

Hence $-2C+8D=0 \to C=4D \to C = -\frac{4}{65}$

$16C+D=-1 \to 64D+D=-1 \to D=-\frac{1}{65}$

$$\mathcal{L}^{-1}\left(\frac{2s^2+s+8}{(s^2-s-12)(s^2+4)}\right)(t) = \frac{11}{35}e^{4t} - \frac{23}{91}e^{-3t} - \frac{4}{65}\cos(2t) - \frac{1}{130}\sin(2t)$$

2. [20] Use Laplace transform to solve the initial-value problem: $\begin{cases} y'' - y = 6e^{-2t} + 4 \\ y(0) = 0; y'(0) = 0. \end{cases}$

Set $Y(s) = \mathcal{L}y(s)$

$$\mathcal{L}(y''-y)(s) = \mathcal{L}(6e^{-2t}+4)(s) = \frac{6}{s+2} + \frac{4}{s}$$

$$\mathcal{L}(y'')(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

$$\mathcal{L}(y''-y)(s) = s^2Y(s) - Y(s) = (s-1)(s+1)Y(s)$$

Thus $$Y(s) = \frac{6s+4(s+2)}{s(s+2)(s-1)(s+1)}$$

$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1} + \frac{D}{s+1}$$

$$= \frac{A(s+2)(s^2-1) + Bs(s^2-1) + Cs(s+2)(s+1) + Ds(s+2)(s-1)}{s(s+2)(s^2-1)}$$

So $$A(s+2)(s^2-1) + Bs(s^2-1) + Cs(s+2)(s+1) + Ds(s+2)(s-1) = 10s+8$$

For A, set $s=0$: $-2A = 8 \to A = -4$

B, set $s=-2$: $-6B = -20+8 = -12 \to B = 2$

C, set $s=1$: $6C = 18 \to C = 3$

D, set $s=-1$: $2D = -2 \to D = -1$

Hence $$Y(s) = -\frac{4}{s} + \frac{2}{s+2} + \frac{3}{s-1} - \frac{1}{s+1}$$

and $$y(t) = \mathcal{L}^{-1}(Y)(t) = -4\mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) + 2\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)(t) + 3\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)(t) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)(t)$$

$$= -4 + 2e^{-2t} + 3e^{t} - e^{-t}$$

3. [20] Find the Laplace transform $\mathcal{L}h$ if $h(t) = \begin{cases} 5, & 0 \le t < 4, \\ -3, & 4 < t < 6 \\ t, & t > 6. \end{cases}$

$$h(t) = 5(1 - u_4(t)) - 3(u_4(t) - u_6(t)) + t u_6(t)$$
$$= 5 - 5u_4(t) - 3u_4(t) + 3u_6(t) + t u_6(t)$$
$$= 5 - 8u_4(t) + 9u_6(t) + (t-6)u_6(t)$$

Hence

$$\mathcal{L}h(s) = 5\mathcal{L}(1)(s) - 8\mathcal{L}(u_4)(s) + 9\mathcal{L}(u_6)(s) + \mathcal{L}((t-6)u_6)(s)$$
$$= \frac{5}{s} - 8\frac{e^{-4s}}{s} + 9\frac{e^{-6s}}{s} + \frac{e^{-6s}}{s^2}$$

4. [20] Find two linearly independent power series solutions of the differential equation: $y'' + xy' - 3y = 0$. Include in each series the powers of x up to x^4. Write down the general solution of the differential equation.

Seek $y(x) = \sum_{n=0}^{\infty} c_n x^n$. So $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Reporting in D.E, we find:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - 3\sum_{n=0} c_n x^n = 0, \text{ for all } x$$

in some interval $|x| < R$, $R>0$

$\downarrow$ Replace $n-2$ with n

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} (n-3) c_n x^n = 0$$

Note that $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$, nothing is added, and all powers of x are nonnegative; this explains why we've gathered the last two sums.

Hence $$\sum_{n=0}^{\infty} \left((n+2)(n+1) c_{n+2} + (n-3) c_n\right) x^n = 0, \text{ for all } x \text{ as above}$$

so
$$(n+2)(n+1) c_{n+2} + (n-3) c_n = 0, \quad n = 0, 1, 2, 3, \dots$$

$n=0$: $2c_2 - 3c_0 = 0 \rightarrow c_2 = \frac{3}{2} c_0$

$n=1$: $6c_3 - 2c_1 = 0 \rightarrow c_3 = \frac{1}{3} c_1$

$n=2$: $12 c_4 - c_2 = 0 \rightarrow c_4 = \frac{1}{12} c_2 = \frac{1}{12} \cdot \frac{3}{2} c_0 = \frac{1}{8} c_0$

$n=3$: $20 c_5 = 0 \rightarrow c_5 = 0$; so $c_7 = c_9 = \dots = c_{2p+1} = 0$ for all $p \geq 2$

Thus, the general soln is

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$
$$= c_0 \left(1 + \frac{3}{2} x^2 + \frac{1}{8} x^4 + \dots\right) + c_1 \left(x + \frac{x^3}{3}\right)$$

c_0, c_1 = arbitrary constants

$y_1(x) = 1 + \frac{3}{2} x^2 + \frac{1}{8} x^4 + \dots$ and $y_2(x) = x + \frac{x^3}{3}$ are linearly independent solutions. Notice that y_2 is a polynomial function.